Liquidity Insurance and Pledgeability *

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Abstract

I examine the optimal design of liquidity insurance contracts for firms that experience a quality shock concurrent with the liquidity shock, both of which cannot be verified and hence contracted upon. Due to the incompleteness of these contracts, firms cannot receive full liquidity insurance: If a firm is fully insured, it has little incentive to stop inefficient projects, as creditors bear the costs. Therefore, the optimal contract involves limited insurance and requires co-investment with internal cash. Interestingly, my findings challenge the classical theory that low-pledgeability firms rely more on liquidity insurance. Instead, I show that a lack of pledgeability prevents these firms from obtaining more liquidity insurance. In fact, I demonstrate a positive relationship between liquidity insurance and pledgeability, which sheds light on the seemingly paradoxical fact that smaller firms, who need liquidity insurance the most, are the least insured and face the highest risk of revocation. Furthermore, my results rationalize the common cash-related covenants in credit lines.

1 Introduction

Credit lines are one of the most important tools of corporate liquidity management besides cash and debt issuance. Bank lines of credit, including committed but not used portion,

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account for over 70% of firms' sources of credit (Greenwald et al., 2021). The classical theory (Holmström and Tirole, 1998) motivates the use of credit line as an ex-ante contract to provide liquidity insurance when a firm would be otherwise unable to issue debt upon the shock due to lack of pledgeability, implementing a transfer of liquidity from the not-shocked state to the shocked state. One of the most highlighted properties of credit lines is its commitment: the creditor couldn't break even conditional on the realization of liquidity shock[1], and would be willing to renege on the contract whenever they can. Therefore, to provide insurance, a credit line must be committed. However, recent empirical papers find that most credit lines are not unconditional.[2] They are often coupled with restrictive covenants on debt and cash, including coverage ratio, leverage, and other financial ratios, violating which would lead to the revocation of credit access. And a significant number of firms do lose credit access in the real world. This seems to suggest the liquidity insurance view is incorrect (or at least incomplete): after all, credit lines are not always committed. But then it's perplexing why firms would pay a premium for a promise not committed. Indeed, these papers mostly focus on the trade-off between creditors' discretion and commitment, while the exact content of the covenants is usually overlooked. In this paper, we want to argue that instead of invalidating the liquidity insurance views, the presence of these covenants, especially the more puzzling ones on cash, as pointed out by Sufi (2009), are actually meant to solve the incentive problem that comes with liquidity insurance. They provide better insurance.

In this paper, I model the provision of liquidity insurance when the firm with limited pledgeability is privately informed about the quality of the firm. I start with a financially constrained entrepreneur who needs to raise external capital to finance a long-term project subject to a liquidity shock and limited pledgeability. Lack of pledgeability prohibits the firm from raising capital at the interim stage, even if the original debt can be diluted. Due to the financial constraint, the entrepreneur has to issue debt to fund the project and cannot keep enough cash reserve to mitigate the liquidity shock since keeping an excessive cash buffer is costly when the liquidity shock does not occur. A more efficient way to mitigate the liquidity shock is to obtain liquidity insurance: the firm pays the lender a premium ex ante in exchange for a promise to provide liquidity when the shock occurs. This implements a state-contingent transfer from the not-shocked state to the shocked state. And since the lender only pays the liquidity cost when the shock occurs, the premium is less expensive than hoarding cash.

[1]If the creditor is able to recoup the funding, it would not be necessary for the firm to pay the premium to maintain the access to the credit as the lender would be willing to do so ex post.

[2]Most prominently Sufi (2009), Chodorow-Reich et al. (2022),Greenwald et al. (2021), and Acharya et al. (2020).

However, full liquidity insurance is no longer efficient when the firm privately learns the quality of the project. Imagine a firm knows that the quality of the project is bad: the continuation value in expectation is lower than the liquidity cost. (I.e., it has a negative interim NPV.) If he has a deep pocket and pays the liquidity cost himself, he would terminate the project efficiently. But since the liquidity cost is borne entirely by the lender, the firm would have the incentive to continue the project regardless of the project quality as the limited pledgeability guarantees him a strictly positive payoff from continuation. It is worth noticing that the lender would have anticipated the inefficient continuation and excessive drawdown and priced it at the cost of liquidity insurance. Therefore, it is the entrepreneur himself who bears the cost of his inability to commit.

This inefficient continuation problem can be solved by requiring the firm to keep a certain amount of cash reserve for co-investment. The simple intuition behind the design is to let the firm have enough skin in the game. It's just like the copay, which is ubiquitous in health insurance. What the lender sells to the firm is basically an option to obtain liquidity but at a certain strike price. When the entrepreneur decides whether to continue the project or not, he will trade off his private payoff (the unpledgeable part of the total payoff) against his cost of co-investment, the cash he could divert. With a calibrated "strike price", when the project is inefficient, i.e., the expected present value of future cash flow is lower than the liquidity cost, the expected present value of the firm's private payoff is also lower than the cash co-investment (i.e., the "strike price") and the entrepreneur optimally chooses not to continue the project as the liquidity option is out of money. This co-investment has been verified in the data. Berrospide and Meisenzahl (2015) finds that

> *"[...] a one standard deviation increase in drawdowns reduces average cash holdings by about 1 percent of total assets [...] reduce cash holdings by about 8 percent [...]"*

which nullifies the precautionary drawdown and can only be explained by a co-investment mechanism.

In order to calibrate the cash requirement, we need to compare what the firm gets in continuation with the "strike price" of liquidity. If the firm has a high pledgeability, which implies the creditor will get a big chunk of the payoff and the firm a small one, it only needs to bear a small fraction of the cost to align the incentive. Thus, a high-pledgeability firm can afford a higher level of liquidity insurance.

The paper highlights an overlooked role of cash that divertable cash can serve as *contingent* skin-in-the-game, that is, it's embedded an option to be converted to skin-in-the-game *ex interim*, which discipline firms' action *ex interim*, not *ex post*.

I then turn to microfounding pledgeability using moral hazard in order to explain the cross-sectional heterogeneity we observed in the real world: larger firms are better insured. They typically have a higher credit limit and less restrictive covenants, and their liquidity access is revoked less often ex post, as documented in Chodorow-Reich et al. (2022). I model a large firm as one that has multiple projects. The entrepreneur can choose, for each project, independently to exert efforts and boost the probability of success or shirk and obtain a private benefit. In order to incentivize the entrepreneur to exert effort, he must be compensated fairly, which implies that a fraction of the payoff cannot be pledged to the creditors. The pledgeability is higher when the moral hazard problem is less severe. Having multiple projects allows the entrepreneur to alleviate the moral hazard problem à la Laux (2001). To see why, suppose the firm has only one project, and in order to be incentivized to exert effort, there must be a discrepancy in the firm's payoff between when the project succeeds and fails. However, due to the limited liability protection of the firm, the lowest possible payoff when the project fails is zero. It would be less costly to incentivize the firm to exert effort if it can be punished more when the project fails. With multiple projects, this is possible. Imagine the firm has two projects. Now when one project succeeds, and the other fails, the firm can be punished for the failed project by stripping away the rewards of the successful project. This cross-subsidization allows the firm to be incentivized at a lower cost, which boosts the pledgeability of the firm and hence makes providing more liquidity insurance to the firm feasible.

To be more specific about the mechanism, let's take a look at a firm with two projects. For the purpose of exposition, let's assume the quality of one of the projects is fixed and is good[3]. (So this project should be continued regardless.) We now show that we can obtain a stronger result: the firm can be fully insured against the liquidity shocks of both projects.[4] Now, suppose the entrepreneur is contemplating whether to continue the second project or not. Continuing an additional project has two effects: i) it increases the total payoff available, conditional on success, incentivizing continuation; ii) it also increases the pledgeability, discouraging continuation. And these two effects are different for projects of different types. When the project is bad, the pledgeability increases less than that with a good project, but the probability of getting paid is also less as the bad project has a lower probability of success. It turns out that the second effect always dominates the first, which means that whenever the entrepreneur wants to continue a bad project, he also wants to continue a good one, but not vice versa. However, this doesn't imply

[3]Indeed, it could even be a zero interm-NPV project. The result would obtain as long as it can facilitate the cross-subsidization mechanism and relax the limited liability constraint of the entrepreneur.

[4]The discussion above would only imply that the large firm with two projects can be *better* insured but not *fully* insured, as the firm would still want to continue two bad projects if it's fully insured. Here, we require one project to be good (or at least zero interim-NPV), so when fully insured, the firm can always continue this project, and we will show continue another bad project is strictly worse.

the entrepreneur always wants to continue one more project. Adding one additional project increases the amount of private benefit to be compensated for but also lowers the probability of being paid. The latter is more salient for the bad project so the entrepreneur might only want to continue a good project. We will show in Section 6 that this is the case when the quality differential, the difference in the probability of success between the good and bad projects, is larger than the effort differential.

The baseline model answers Sufi (2009)'s question of why credit lines, a substitute for cash, a tool that is meant to solve the low cash problem and provide liquidity, require the firm to have cash in order to draw down. The answer is that the covenant aims to solve the moral hazard problem of inefficient continuation brought about by the insurance. And firms with higher pledgeability have a less severe incentive problem and can be better insured. This positive relationship explains the commonly observed facts that large firms, which typically have higher pledgeability, are way better insured than the smaller ones in terms of credit limit, covenants presence, and revocation.

In what follows, I will first briefly introduce the institutional detail and guide the readers through the relevant literature in Section 2. Then I set up the baseline model in Section 3 and present the main results along with many benchmarks in Section 4. To extend the model to firms with multiple projects and investigate the relationship between liquidity insurance and pledgeability, I will first microfound the pledgeability in Section 5 and then discuss the continuation choices of the firm in Section 6. Other extensions are considered in Section 7. Section 8 concludes the paper.

2 Institutional Details and Literature Review

Institutional Details Before proceeding to the model, let's pause for a while and look at these covenants. Most covenants in the contract of lines of credit are concerned with firms' cash flow and leverage: they require firms to keep a certain level of cash and not borrow too much. Sufi (2009) finds that about 75% of the lines of credit that have financial covenants have cash flow-based covenants (see his Table 5)[5], for example, in the form of coverage covenants, debt to cash flow covenants, etc. Coverage covenants specify a lower bound of the ratio of cash flow to interest expenses, fixed charges, and debt services. Acharya et al. (2020), in the working paper version, finds more than 30% of the covenants are cash flow related.[6] (Note that lines of credit containing multiple covenants are counted multiple

[5]His sample ranges from 1996 to 2003, including 11,758 loan deals by 4011 public firms found in Dealscan

[6]Their sample includes 22,680 covenants found in the loans of non-financial firms from 1995 to 2011 in the merged Capital IQ-Compustat and Dealscan database.

times.) They most commonly concern interest coverage and fixed charge coverage.

This seems to be contradictory to the purpose of having a credit line. Firms use credit lines to mitigate the problem of insufficient cash, so they used to be considered a substitute for cash. By requiring the firm to have enough cash to avoid revocation, a credit line is not a perfect substitute for hoarding cash. Similarly, the commitment is meant to give firms access to credit when they cannot borrow ex post due to limited pledgeability. Hence, the creditor shouldn't be worried about the firm borrowing too much. In addition, many credit lines with covenants are usually secured and shouldn't be too afraid of dilution. Therefore it's not trivial why the creditors condition the access of credits on cash and debt.

Now I will briefly go over the paper's connection and contribution to the literature, both theoretical and empirical.

Theoretical literature on (liquidity) insurance The paper contributes directly to the literature on liquidity insurance following the seminal paper by Holmström and Tirole (1998), which established that firms with low pledgeability need to rely on liquidity insurance. In contrast to their conclusion, I find that a lack of pledgeability can also prohibit firms from obtaining liquidity insurance due to the incentive to continue inefficient projects. This main mechanism of the paper echoes the theme in Rampini and Viswanathan (2018) that insurance is expensive to finance, and not everyone can afford it, especially the ones that are financially constrained. Another paper that directly speaks to the issue of liquidity insurance is Acharya (2014). They address another moral hazard issue with liquidity insurance that the firm could substitute the liquid project with an illiquid project after obtaining the liquidity insurance, i.e., the firm could increase the probability of the liquidity shock, which of course, yields a higher payoff. The cost is also borne by the lenders in their model as the probability of the drawdown cannot be contracted upon. This paper differs from theirs in that we focus on the moral hazard in the interim continuation decision instead of manipulating the probability of liquidity shock, which is similar to the commonly studied asset substitution problem.

Theoretical literature on covenants Donaldson et al. (2019) is the most similar to mine in the sense that the role of the covenant is to allow the privately informed party to reveal private information through covenant violation. Moreover, covenants are meant to solve the inability of committing not to undertake inefficient action. Differently, Donaldson et al. (2019) focus on the covenants related to future secured lending in unsecured debt, while mine is about the cash covenants in the credit line. Also, covenants violations are always waived in their model but not in mine.

Rajan and Winton (1995) is one of the earliest papers discussing the role of covenants and highlights the function of covenants in implementing the contingent transfer of control à la Aghion and Bolton (1992): the control is transferred from the borrower to the creditor when the covenant is violated. But covenant violation is exogenous in their model, and this paper augments this view by allowing the firm to violate the covenants strategically.

In contrast to debt covenants, there are very few theories on credit line covenants. And most of them treat credit line covenants at creditors' discretion without exploring the exact contents of the covenants. Chodorow-Reich et al. (2022) assume the bank's discretion when the credit line is not committed. The bank has the option to observe the project's quality and decide whether to honor the credit line, and the bank is only willing to do so when it can break even *ex post*. In other words, the credit line doesn't serve the role of liquidity insurance. However, this model misses an important aspect of credit line revocation: banks can revoke the credit line only if the firm fails to comply with the covenants, which are at the discretion of the firm. Also, the discretionary credit lines do not provide liquidity insurance in their model.

Empirical literature on covenants in the credit lines The empirical evidence documenting the stringent covenants requirement starts from Sufi (2009), who points out that, paradoxically, many bank lines of credit come with cash-flow-based financial covenants, which prescribe firms to maintain a high current ratio or, a low payout ratio or other types liquidity related covenants. Failure to comply with these covenants could lead to a 15% to 25% drop in total and unused lines of credit. In his samples, roughly 85% of firms have a line of credit that provides, on average, 50% of firms' total liquidity, and about 35% of the firms have committed a covenant violation. He also finds that the only variable predicting a covenant violation is the leverage ratio, besides a drop in cash flow. Despite his emphasis on cash flow, these covenants can also be alternatively interpreted as covenants on the amount of existing debts before a drawdown. Similar evidence is also presented in Acharya et al. (2020). In their samples, about 30% of covenant occurrences related to leverage (e.g., Debt to EBITDA, leverage ratio) and around 35% cash-flow related (e.g., Interest Coverage, Fixed Charge Coverage, and EBITDA.) Davydenko et al. (2020) argues in term loans, covenants for not useful for highly levered firm and the triggering of technical defaults leads to lower payoffs for the creditor.

3 Model Setup

The baseline model involves a financially constrained entrepreneur who has wealth w and tries to fund a project that generates a stochastic return and is subject to liquidity shock at the interim stage. The entrepreneur faces a competitive lending market and can use various contracts to secure funding. Crucially, the entrepreneur (and his effort in Section 5)[7] is indispensable for the project so he cannot simply sell the project to the investors.

There are three dates $t \in \{0, 1, 2\}$, which we will respectively call ex-ante, ex-interim, and ex-post. At date 0, the entrepreneur seeks funding and signs certain contracts with creditors. Contracts are not restricted to being exclusive, so the entrepreneur is allowed to contract with multiple lenders. The non-exclusivity would allow the entrepreneur to raise capital at the interim stage, diluting the existing creditors. But as we will see, the non-exclusivity won't play an important role as there is not enough pledgeability to raise capital at the interim stage, so we can focus our attention on the case where the entrepreneur only borrows from one investor.

The project, once founded, might need a maintenance cost $\tilde{\rho} \in \{0, \rho\}$ unobservable to the outsiders at date 1. So it cannot be contracted upon. The mismatch of the long-term asset and short-term liquidity shock and the contractual incompleteness is the element from Holmström and Tirole (1998). We say the project is "shocked" whenever $\tilde{\rho} = \rho$ and "not shocked" otherwise. The project is liquidated at a residual value 0 if the maintenance cost is not paid. We allow for partial liquidation, which we dub "downsizing", if a fraction x of the liquidity cost (i.e.,$x\rho$) can be paid, but as a consequence of downsizing, the payoff upon success is only xR.[8] If the project is continued, it yields a payoff R with probability p_Q where Q is the quality of the project. The quality can be either good ($Q = G$) with probability $q = \Pr(Q = G)$ or bad ($Q = B$) with the complementary probability $1 - q$, and it is only revealed to the entrepreneur at the beginning of date 2. As a consequence, the contract also cannot be contingent on the quality of the project. This assumption is absent in Holmström and Tirole (1998) and leads to the paper's main result that liquidity insurance is limited by pledgeability. [9] With a little abuse of notation, we denote $p_q := qp_G + (1-q)p_B$

[7]We use "he/him/his" for the entrepreneur and "she/her/hers" for a single creditor/investor.

[8]Downsizing is a standard assumption in the dynamic contracting literature, e.g., in Biais et al. (2010). Here this assumption allows us to have a "variable" liquidity shock size in the binary framework. This assumption will enable us to see why a simple covenant on the liquidity shock is insufficient and why an explicit mention of cash must be made.

[9]In Holmström and Tirole (1998), the distribution of the liquidity shock is continuous. So there is a notion of qualities: the project is good if the maintenance cost is lower than the present value of the cash flow. But the entrepreneur is only able to invest in good projects since the credit line (the optimal contract in HT98) has a cap. This does not work here, as both good and bad projects have the same maintenance costs.

the ex ante expected quality of the project.

The entrepreneur is allowed to interact with the financial markets at the interim state $t = 1$ to mitigate the liquidity risk but only a fraction θ of the date 2 cash flow R. This motivates liquidity insurance.

Another important institutional detail is the diversion of cash, which can be in the form of dividend payments, executive compensation, etc. We assume that the entrepreneur can divert cash without legal consequences *before or after* he learns about the project quality and liquidity shock at date 1, but only *before* drawing down the credit line (if used in equilibrium). Diverting cash before learning everything is effectively reducing w. This means the entrepreneur prefers to use more debt financing than internal wealth. We will see that in equilibrium, the entrepreneur has an incentive to do so in order to reduce his incentive to continue. The second element we embed here is that with existing debt, the entrepreneur can only divert cash when the default is not imminent, i.e., he can do so at date 1 but not date 2. This is consistent with the current US bankruptcy code. [10] In addition, the consequences of such transfers are different before and after drawing down the line of credit. If the diversion occurs before the drawdown, it might violate some covenants and trigger a technical default, leading to the revocation of the line, but usually, the creditor won't be able to avoid the transfer. However, it is different after the drawdown. There is usually a restriction on the use of proceeds, and there are additional "springing covenants" which become effective when the borrower utilizes the revolving facility[11]. For example, in Nordstrom's *Revolving Credit Facility Agreement* [12], the use of the proceeds are strictly specified:

> **Section 2.3. Use of Proceeds.** The proceeds of the Loans shall be used by the Borrower only for working capital, capital expenditures and other lawful general corporate purposes of the Borrower and its Subsidiaries, including (a) loans made by the Borrower to its Subsidiaries and (b) the payment of commercial paper [...] the proceeds of the Loans shall not be used to finance any acquisition [...]
>
> **Section 6.2. Restricted Payments.** The Borrower shall not, and shall not permit any Subsidiary to, declare, pay or make, or agree to declare, pay or make, any Restricted Payment [...] ("Restricted Payment" includes (i) any dividend or other distribution [...])

[10] Typically, such transfers are deemed fraudulent if they are made up to two years before filing the petition and preferential if it is made up to 90 days before the filing to a creditor. (See 11 U.S. Code § 544, 548, and 547.)

[11] They are also popular among syndicated loans. See Ivashina and Vallee (2020), Berlin et al. (2020).

[12] See the full agreement on the SEC website `https://www.sec.gov/Archives/edgar/data/72333/000007233318000148/jwn-10022018xex101.htm`

Therefore, in accordance with the law and the practice, the entrepreneur will not be allowed to draw down the credit line and then divert the cash.

Parametric Assumptions To focus on the most interesting cases, we impose several parametric assumptions in the baseline model.

Assumption 1 (Ex-Post Efficiency)**.** *The liquidity shock is neither too small nor too large*

$$p_B R < \rho < p_G R \tag{1}$$

This assumption says that the liquidity cost is only worth paying if the project is good. Continuing a bad project is not ex-post efficient.

Assumption 2 (Ex-Ante Efficiency)**.**

$$NPV^{FB} := q p_G R + (1-q)(1-\lambda) p_B R - I - q\lambda\rho > 0. \tag{2}$$

This assumption ensures that the project has a positive NPV if it's carried out and if it's discounted only when there is a liquidity shock and the quality turns out to be bad. We do not take a stance on whether the project is worth investing in when the ex-post continuation-liquidation decision is not optimal for now. These two assumptions allow us to define the notion of the first best

Definition 1 (First Best)**.** *The project achieves the first best if it is invested and continued if i) there is no liquidity shock or ii) if there is a liquidity shock and the quality is good.*

Assumption 3 (Limited pledgeability)**.** *The expected pledgeable cash flow is less than the liquidity shock even when the project is good*

$$p_G \theta R < \rho \tag{3}$$

This renders interim-stage funding impossible and motivates the need for ex-ante contractual arrangements to provide liquidity insurance. Otherwise, the entrepreneur could just wait until the liquidity shock occurs and issue senior claims by collateralizing the future cash flow, diluting the existing creditor.

Assumption 4 (Limited Wealth)**.**

$$\underline{w}^{Cov} \leq w < I + \rho \tag{4}$$

where

$$\underline{w}^{Cov} := I + q\lambda\rho + (1 - q\lambda)p_B(1 - \theta)R - [qp_G + (1 - q)(1 - \lambda)p_B]\theta R \tag{5}$$

The second inequality in the limited-wealth assumption motivates the lending in the first place, i.e., the firm cannot entirely fund the project by itself, which leads to the distorted incentive to continue in the interim stage due to the existence of the debt. Without this assumption, as we will show in Section 4, a deep-pocketed entrepreneur can achieve the first best. The first inequality ensures the firm has enough wealth to borrow, obtain a credit line, and keep some cash reserves, as we will show in Proposition 4.

Notice that

$$I + \rho - \underline{w}^{Cov} = (1 - q\lambda)[\rho - p_B(1 - \theta)R] + [qp_G + (1 - q)(1 - \lambda)p_B]\theta R > 0 \tag{6}$$

where the first term is positive given Assumption 1, so the region is never empty.

Following the convention, we assume creditors are competitive, and all agents are risk neutral. The solution concept we will employ is Perfect Bayesian Equilibrium (PBE), which requires all strategies to be sequentially rational and all beliefs are consistent on path. Whenever the agent is indifferent between two choices, we assume he chooses the socially desirable outcome.

4 Benchmarks and Main Results

In this section, we examine various implementations commonly used in practice. We start by comparing several benchmarks, which aim to show in the presence of noncontractable quality and liquidity shock, the conventional tools, such as interim credit market, internal cash holdings, and unconditional lines of credit, cannot achieve first best. Then, we show a credit line with cash covenants can achieve the first best.

To begin the discussion, we first show that the first best can be achieved when the firm has abundant internal wealth so its decisions, both ex ante and ex post, are not distorted by other financial contracts.

4.1 Benchmark I: Deep-pocketed investor achieves first best

We first show that if the Assumption 4 is relaxed, i.e., the entrepreneur has enough to internal wealth to implement the project and satisfy the liquidity shock, the first best can be achieved.

Lemma 1. *If $w \geq I + \rho$, the following action profile constitutes a PBE:*

1. *The entrepreneur incurs initial cost I to the fund the project at $t = 0$.*

2. *At $t = 1$, if the project is shocked ($\tilde{\rho} = \rho$) and if the quality is good ($Q = G$), the entrepreneur pays the liquidity cost ρ and continues the project; if the project is not shocked, the entrepreneur always continues the project no matter the quality.*

And the first best is achieved in this equilibrium.

The first best can be obtained because the cost of the continuation is fully internalized, and the entrepreneur has a full stake in the project, so the continuation decision would not be distorted by the existence of other claimants (as there are none). But, implementing this equilibrium requires an abundance of wealth, which cannot be achieved when the entrepreneur is financially constrained.

In what follows, I consider a series of tools commonly used in corporate finance and discuss whether they can achieve first best.

4.2 Benchmark II: Interim market does not achieve first best

Now let's consider a financially constrained entrepreneur who cannot carry out the first best implement himself but is free to go on the spot market to finance the project at both $t = 0$ and $t = 1$. Only a plain debt contract is considered, and the creditors are competitive, so they earn zero profit in expectation. In particular, we allow the entrepreneur to raise capital from multiple creditors so he can borrow senior debt, diluting the initial creditor when the liquidity shock occurs. This maximizes the interim debt capacity at the expense of ex-ante debt capacity.I.e., knowing they will be diluted at date 2, creditors are not willing to lend much at date 0. This could work if the firm can borrow enough from the spot market, and the dilution can serve as a contingent transfer from the not-shocked state to the shocked state, but not in our case.

Lemma 2. *The firm cannot achieve the first best by borrowing in the spot credit market at date* 1 *when the liquidity shock occurs.*

To highlight the intuition, imagine the firm borrows from an external creditor, he will need to pay at least ρ in expectation to the external creditor, but the maximum credible payment is $p_G(1-\theta)R$ even when the project is good by Assumption 3. So the firm cannot raise enough capital at date 1 to combat the liquidity shock. Similarly, the firm cannot borrow from existing creditors either. The existing creditor may not need to break even on the new lending if he can be compensated enough from the recovery of the initial debt. But this is not feasible. Due to the limited pledgeability. So no internal funding is possible.

This is the same result as in Holmström and Tirole (1998): limited pledgeability reduces the interim debt capacity, and the firm is unable to fund a project when the project is good. The interim spot market leads to too much inefficient liquidation.

4.3 Benchmark III: Cash hoarding does not achieve first best

Another common way for the firm to mitigate liquidity shock is to have enough liquidity buffers by holding cash. But we will show that cash hoarding is also not able to achieve the first best in our model.

Lemma 3. *The firm cannot achieve the first best with cash hoarding $c = \rho$ if either one of the following conditions is met*

1. $\rho > (1-\theta)p_G R$;
2. $D_0 < R - \frac{\rho}{p_B}$ *if condition 1 is not met;*
3. $w < \underline{w}^{Cash} := I + \rho - [qp_G + (1-q)(1-\lambda)p_B]\theta R$ *if both conditions 1 and 2 are not met.*

This result highlighted that the use of debt dilutes the inside equity so the firm may have distorted incentives for the continuation decision. Here there are three possible reasons for failing to achieve the first best: i) The shock size might be larger than the firm's insider equity share of the payoff, thus resulting in an inefficient liquidation. The firm might prefer to run away with the cash; ii) The debt might be too low, so it gives the firm too much from continuing a bad project, leading to inefficient continuation; iii) Despite that the firm's incentive is ex post aligned, keeping cash reserve is costly and tightens firms' wealth constraint ex ante, thus limiting the initial capital the firm can raise from the creditor initially. Indeed, we can show the last threshold is always higher than that for a credit line with a cash covenant.

So far, we have not taken a stance between the size of liquidity shock and the unpledgeable part of the cash flow, i.e., whether condition 1 is met or not, but it turns out, as we will show in Proposition 1, regardless of this condition, internal cash hoarding does not achieve first best for some parameter region of w. This contrast highlights the difference between internal liquidity and *contingent* external liquidity. Internal liquidity is also subject to incentive distortion when there are other claim holders, while contingent external liquidity can be calibrated to eliminate this concern.

To be more precise, we have

Proposition 1. *There exists some w for which cash hoarding cannot achieve first best, but a credit line with cash covenants can.*

The key to the proof is to show that $\underline{w}^{Cash}$ is higher than $\underline{w}^{Cov}$, so for any value $w \in (\underline{w}^{Cov}, \underline{w}^{Cash})$, cash hoarding cannot achieve first best even if the other first two conditions are not met. Therefore, when w is large enough, the first best might be achieved by both cash hoarding and credit line with cash covenants (see Proposition 4), but it is dominated by the credit line with cash covenants when w is not large enough.

4.4 Benchmark IV: An unconditional credit line does not achieve first best

Now we introduce the possibility of obtaining a line of credit, which promises the entrepreneur a chance of obtaining funding at date 1. As Holmström and Tirole (1998) points out, the lender cannot break even ex post, so the provision of liquidity must be unconditional and not renegotiable. Otherwise, the lender always has an incentive to revoke the credit line and reject the funding request. The inability of commitment makes the credit line useless, and the entrepreneur is not willing to pay a fee to maintain the line of credit. However, we will show below that such unconditional full liquidity insurance is sub-optimal.

Proposition 2. *Suppose the entrepreneur obtains a line of credit $\ell = \rho$ from the creditor. In no equilibrium is the first best obtained.*

Furthermore, if $[qp_G + (1-q)p_B]R < I + \lambda\rho$ or $w < \underline{w}^{CL} := I + \lambda\rho - [qp_G + (1-q)p_B]\theta R$, the project is not funded; otherwise, it can be funded with any debt of face value $D_0 \in \left[\frac{I-w+\lambda\rho}{qp_G+(1-q)p_B}, (1-\theta)R\right]$ but is always continued even when it is bad.

This result highlights the problem of full insurance: it encourages too much inefficient continuation. The intuition behind this result is the entrepreneur's inability of committing not to continuing a bad project when he is fully insured. The entrepreneur bears no cost and enjoys at least partially the unpledgeable part of the payoff.

Note here the threshold can also be written as

$$\underline{w}^{CL} = I + \lambda\rho - (1-q)\lambda p_B\theta R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{7}$$

and we have

$$\begin{aligned}\underline{w}^{CL} - \underline{w}^{Cov} &= \lambda(1-q)[\rho - p_B\theta R] - (1-q\lambda)p_B(1-\theta)R & (8)\\ &= \lambda(1-q)[\rho - p_B R] - (1-\lambda)p_B(1-\theta)R & (9)\end{aligned}$$

which can be negative if ρ is sufficiently close to $p_B R$. Therefore when $\underline{w}^{CL} < \underline{w}^{Cov}$, for

$w \in (\underline{w}^{CL}, \underline{w}^{Cov})$, the firm may not be able to achieve first best even with a credit line with cash covenants, but it may nevertheless fund the project via a plain unconditional credit line provided it's profitable.

Renegotiation The inefficient over-continuation problem can be avoided if the lender can ex post bribe the firm not to continue the project by paying him a fee of at least $p_B \max\{(1-\theta)R, R - D_0\}$. This could be implemented without knowing whether the true project quality as the firm with good projects would decline the offer, draw down the credit line and continue the project. However, three issues are involved: i) first, it must be credible that the lender can ensure this bribe works. This may be difficult if the credit line is still available. There's no guarantee to the lender that the firm will terminate the inefficient project and not draw down the credit line. This could be solved if the initial contract specifies that the lender could terminate the contract by paying a severance fee; ii) the lender must be incentive compatible with bribing. Nevertheless, we show that it is possible.

Proposition 3. *If the lender could credibly bribe the firm not to draw down the credit line, the project can be funded with an unconditional credit line.*

4.5 Main Result I: Credit line with covenants on cash achieves the first best

Now we proceed to exhibit our first main result that a credit line with a covenant on cash achieves the first best.

Proposition 4. *The following action profiles and beliefs consist of a PBE, and the first best is achieved in the PBE:*

- *At date $t = 0$, the entrepreneur borrows $I + p_B(1-\theta)R - \underline{w}^{Cov}$ with a debt of face value $D_0 = \theta R$ and chips in his own capital $\underline{w}^{Cov}$ where*

  $$\underline{w}^{Cov} = I + q\lambda\rho + (1-q\lambda)p_B(1-\theta)R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{10}$$

  as defined in Assumption 4. In addition, he obtains a credit line $l = \rho - (1-\theta)p_B R$ from a lender with the cash covenant that he could draw down the credit line only if he maintains a cash reserve of $(1-\theta)p_B R$. The entrepreneur keeps a cash reserve $c = (1-\theta)p_B R$

- *At date $t = 1$, if the liquidity shock does not occur $\tilde{\rho} = 0$, the entrepreneur pays a cash dividend $c = p_B(1-\theta)R$ to himself*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is good $Q = G$, the entrepreneur does not divert cash and draws down the line of credit, and he uses the funds to continue the project.*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is bad $Q = B$, the entrepreneur diverts cash and the lender revokes the credit line. The project is liquidated.*

The lender believes that the project is good with probability 1 *if and only if the entrepreneur does not divert the cash.*

The key element in the result is that cash can serve as *contingent* skin-in-the-game. Due to limited pledgeability, the firm retains a certain fraction of the payoff from the success of the project, which is its inside equity. The continuation decision is a trade-off between the payoff of the inside equity, determined by the pledgeability, and the cost of the inside equity, determined by the cash covenant. A good design of credit lines balances the trade-off and assures that the firm only continues when the project is good. The mechanism resembles the skin-in-the-game, which states that a significant stake in the project discipline the manager's action *ex post*. But it's worth noting that here whether the cash is converted to skin-in-the-game is contingent on the private information of the manager. He exercises the option only when he learned that the project is efficient. It's important that the firm can divert the cash. If, instead, the covenant is on an asset that the creditor can seize when the project is terminated, the manager's incentive to continue a bad project is strengthened instead of weakened. The covenant could also be on other marketable securities, but cash is the least costly one to align the incentive without incurring valuation issues.

An immediate result of this proposition is the relationship between pledgeability and liquidity insurance.

Corollary 1. *The credit limit is increasing in pledgeability.*

This is easily seen from the explicit expression of the credit limit.

One may wonder if the cash covenant is really indispensable as the liquid cost ρ is higher than the credit limit l, then the firm has to prepare cash for the rest of the liquidity cost anyway. Therefore, there might not be a need for the explicit mention of cash in the contract. The next result says this is not the case when the firm has the option to downsize the project.

Proposition 5. *A credit line with reduced credit limit $l = \rho - p_B(1 - \theta)R$ but no cash covenants cannot achieve the first best.*

The key to the proof is to show that the entrepreneur will downsize the project and use the available credit line to continue the downsized project. The reason relies on the firm's marginal payoff from the continuation of a marginal fraction of the project. When the liquidity needs exceed the credit limit, the firm bears the entire marginal cost of the project ρ while the marginal cost when the size is smaller than the credit limit is 0. The marginal payoff for the firm from continuing the bad project is $p_B(1-\theta)R$, which is between 0 and ρ. So the firm would optimally downsize the project to the threshold that can be continued purely using the credit line.

Note downsizing won't help in the case with covenants because the firm cannot divert cash and will only benefit the creditors if the firm chooses to downsize the project but doesn't divert the unused cash.

5 Pledgeability with Multiple Projects

So far, we have assumed that the pledgeability is exogenous. This section provides a microfoundation using moral hazard as per Holmström and Tirole (1998). What's different is now we allow the firms to have multiple projects, and we will show that, in general, more projects increase the firm pledgeability. This is an application of the mechanism uncovered in Laux (2001).[13]

We first state the setting: now, after the continuation decision at date 1, the entrepreneur can decide whether to exert effort or not. Not exerting effort reduces the probability of success by Δp, i.e., the type Q project succeeds with probability $p_Q - \Delta p$ instead of p_G, but it also gives the entrepreneur a private benefit b. When there are multiple projects, the entrepreneur makes a decision on effort for each project individually and obtains private benefit b for each project he shirks on. Throughout, we assume effort is always efficient[14] and we define pledgeability to be the maximum share of the return that can be pledged to the creditors without reducing the incentive to exert effort.

Typically, in order for the existing debt contract to be incentive compatible, there is

[13]It is crucial that the same team operates the two projects. If a large firm is bureaucratic, assigning multiple teams to a single project would overturn the result. A related mechanism is used in Boot and Schmeits (2000), where they discussed multiple bank monitoring. Having multiple monitors undermines the incentive to monitor due to the free-riding with perfect information. Surprisingly, the problem is alleviated when the information is private.

[14]That is, even taking entrepreneurs' private benefit into consideration, it's still more efficient to exert effort. For example, in the single-project case, it means $\Delta pR > b$. In the principal-agent relationship, the principal may not always want to make an effort, as the cost of implementing the effort might outweigh the efficiency improvement. Here, it's not the case. The lenders are ex ante competitive, and therefore, the firm captures the full surplus from the effort, and it's always in the firm's interest to implement effort, forgoing the private benefit.

a renegotiation at date $t = 1$ *after* the continuation decision[15], which is well anticipated when the debt contract is assigned. We assume that the outcome of the renegotiation is a minimum write-down in the debt such that the

5.1 Single Project

We start with the case of a single project. The pledgeability of a project of type p_Q is defined as follows

Definition 2 (Pledegability-Single Project)**.** *The pledgeability of the project p_G is the maximum fraction of the return that can be pledged such that the effort is still incentive compatible, i.e.,*

$$\theta(p_Q) := \max\{\theta \in [0, 1] : p_Q(1 - \theta)R \geq (p_Q - \Delta p)(1 - \theta)R + b\} \tag{11}$$

Simple algebra yields $\theta(p_Q) = 1 - \frac{b}{R\Delta p}$. Notice this is a constant function of p_G, so the pledgeability is the same whether the project is good or bad. This is only true for the single-project case.

5.2 Multiple Projects

Now let's consider n projects with quality profiles $\boldsymbol{p} = (p_1, p_2, ..., p_n)^\top$. Also, let $\boldsymbol{e} \in \{0, 1\}^n$ be an effort profile where $e_i = 1$ means the entrepreneur exerts effort on project i while $e_i = 0$ indicates he shirks on the project i. Naturally, $\boldsymbol{e}^\top \boldsymbol{e}$ is the number of projects the entrepreneur exerts effort on. The vector $\mathbf{1} = (1, 1, ..., 1)^\top$ is the effort profile in which the entrepreneur exerts efforts on every project. Similarly, the vector $\mathbf{0} = (0, 0, ..., 0)^\top$ is the effort profile in which the entrepreneur exerts no effort on any projects.

In general, the contracting space is large, and there are many different incentive constraints, but as Laux (2001) points out that i) the optimal contract only pays the agent when all projects succeed, and ii) the tightest constraint is that exerting effort $\mathbf{1}$ is better than $\mathbf{0}$. [16]

[15] The entrepreneur could renegotiate with the lender before the continuation decision, but this will not restrain the continuation choice as long as the entrepreneur cannot commit not to renegotiate again after the continuation decision.

[16] It is worth noting that the optimal contract is feasible if there are enough to compensate the entrepreneur. Otherwise, the investor has to compensate the entrepreneur out of their own pockets when all projects succeed. This is rarely seen in real life, so we maintain the assumption that [...] To be more precise, we assume $\frac{b}{R} < \prod_{i=1}^n p_i - \prod_{i=1}^n (p_i - \Delta p)$. For the contract implementable by debt, we also need $\theta(\boldsymbol{p})nR \geq (n-1)R$, which is a stronger requirement that $\frac{nb}{R} < \prod_{i=1}^n p_i - \prod_{i=1}^n (p_i - \Delta p)$.

Definition 3 (Pledgeability-Multiple Projects)**.** *The pledgeability of the project profile* $\boldsymbol{p}$ *is the maximum fraction of the return that can be pledged such that the effort is still incentive compatible, i.e.,*

$$\theta(\boldsymbol{p}) := \max\{\theta \in [0,1] : (1-\theta)nR\prod_{i=1}^{n} p_i \geq (1-\theta)nR\prod_{i=1}^{n}(p_i - \Delta p) + nb\}. \tag{12}$$

Solving the inequality gives

$$\theta(\boldsymbol{p}) = 1 - \frac{b/R}{\prod_{i=1}^{n} p_i - \prod_{i=1}^{n}(p_i - \Delta p)}. \tag{13}$$

Notice now that pledgeability is no longer a constant function of the project qualities. In particular, we find that better projects have higher pledgeability.

Lemma 4. *Consider two project profiles* $\boldsymbol{p}^{\alpha} = (p_1, ..., p_j^{\alpha}, ..., p_n)^{\top}$ *and* $\boldsymbol{p}^{\beta} = (p_1, ..., p_j^{\beta}, ..., p_n)^{\top}$ *that only differs in element* j*, and* $p_j^{\alpha} > p_j^{\beta}$*, we have* $\theta(\boldsymbol{p}^{\alpha}) > \theta(\boldsymbol{p}^{\beta})$*.*

This result gives us a lattice structure of the pledgeability, and we have, by induction, the following

Corollary 2. *Consider two projects profile* $\boldsymbol{p}^{\alpha}$ *and* $\boldsymbol{p}^{\beta}$ *such that* $p_i^{\alpha} \geq p_i^{\alpha}, \forall i$ *and there is at least one strict inequality, then we have* $\theta(\boldsymbol{p}^{\alpha}) > \theta(\boldsymbol{p}^{\beta})$*.*

This result highlights that the pledgeability is higher when the projects are better.

The next lemma tells whether adding one more project would increase pledgeability:

Lemma 5. *Consider two project profiles* $\boldsymbol{p}^{(n)} = (p_1, ..., p_n)^{\top}$ *and* $\boldsymbol{p}^{(n+1)} = (p_1, ..., p_{n+1})^{\top}$*,* $\theta(\boldsymbol{p}^{(n)}) \leq \theta(\boldsymbol{p}^{(n+1)})$ *if and only if*

$$p_{n+1} \geq 1 - \left(\prod_{i=1}^{n} \frac{p_j}{p_j - \Delta p} - 1\right)^{-1} \Delta p \tag{14}$$

An additional project increases the pledgeability if and only if it's good enough. One may incorrectly conclude that an entrepreneur only wants to continue an additional bad project. This is wrong for two reasons: i) despite the fact that the pledgeability is higher, the total available cash flow is also higher with more projects; ii) the probability of all projects succeeding is also higher when the additional project is good (rather than bad).

So far, the definition of pledgeability only tells us how much share of the cash flow, once all projects succeed, can be taken by the creditor while preserving the incentive of the entrepreneur to exert effort. But the creditors also get paid when not all projects succeed. So it is useful to establish another notion of expected pledgeability, which measure how much can be pledged to the creditors in expectation. In expectation, the projects payoff conditional on efforts profile $\mathbf{1}$ is $\sum_{j=1}^{n} p_j R$ and the entrepreneur is paid at most in expectation

$$\bar{R}_b(\boldsymbol{p}) := (1-\theta(\boldsymbol{p}))nR\prod_{j=1}^{n} p_j = \frac{nb}{1-\prod_{j=1}^{n}\tilde{p}_j} \tag{15}$$

where

$$\tilde{p}_j := \frac{p_j - \Delta p}{p_j}, \tag{16}$$

which measures the percentage loss in the expected payoff due to lack of effort. Note $\tilde{p}_j$ is increasing in the project quality p_j so we could use it as a characterization of the project quality.

Lemma 6. *Consider two project profiles $\boldsymbol{p}^{\alpha} = (p_1, ..., p_j^{\alpha}, ..., p_n)^{\top}$ and $\boldsymbol{p}^{\beta} = (p_1, ..., p_j^{\beta}, ..., p_n)^{\top}$ that only differs in element j, and $p_j^{\alpha} > p_j^{\beta}$, we have $\bar{R}_b(\boldsymbol{p}^{\alpha}) > \bar{R}_b(\boldsymbol{p}^{\beta})$.*

And similarly, this lemma gives rise to the lattice structure.

Corollary 3. *Consider two projects profile $\boldsymbol{p}^{\alpha}$ and $\boldsymbol{p}^{\beta}$ such that $p_i^{\alpha} \geq p_i^{\alpha}, \forall i$ and there is at least one strict inequality, then we have $\bar{R}_b(\boldsymbol{p}^{\alpha}) > \bar{R}_b(\boldsymbol{p}^{\beta})$.*

This result tells us that despite better projects increasing pledgeability, conditional on success, which lowers the entrepreneur's payoff upon success, it also increases the probability of success, and the entrepreneur is paid more often. The latter effect always dominates the former, so the entrepreneur always gets more with better projects even though this leads to a higher pledgeability.

Lemma 7. *Consider two project profiles $\boldsymbol{p}^{(n)} = (p_1, ..., p_n)^{\top}$ and $\boldsymbol{p}^{(n+1)} = (p_1, ..., p_{n+1})^{\top}$, $\bar{R}_b(\boldsymbol{p}^{(n+1)}) \geq \bar{R}_b(\boldsymbol{p}^{(n)})$ if and only if $\tilde{p}_{n+1} \geq 1 - \frac{\prod_{j=1}^{n}\tilde{p}_j^{-1}-1}{n}$, or equivalently*

$$p_{n+1} \geq \left(\prod_{j=1}^{n}\frac{p_j}{p_j-\Delta p} - 1\right)^{-1} n\Delta p \tag{17}$$

This result says that a firm will continue an additional project only if it's sufficiently good. Otherwise, the failure of the additional project will forfeit the payoff from the success of the other projects.

As a special case, we notice that if there are two available projects p_1 and p_2, and if the entrepreneur continues the project 1 for sure, he wants to continue the second project 2 if and only if

$$p_2 \geq p_1 - \Delta p \tag{18}$$

which means that the second project cannot be too bad. In particular, it can not be worse than the first project without effort. This motivates our assumption needed for the next section.

Assumption 5 (Quality Differential)**.** *The quality differential $p_G - p_B$ is larger than the effort differential Δp, i.e.,*

$$\Delta p \leq p_G - p_B \tag{19}$$

This assumption is reminiscent of the single crossing condition commonly used in mechanism design and corporate finance theory.

6 Firm Size and Access to Liquidity Insurance

This section will bridge the gap between pledgeability and liquidity insurance. We model a large firm as one with two projects and a small firm as one with only one project. For the purpose of exposition, we assume one of the projects of the large firm, called Project A, is always good, [17], while the other, Project B, can be either good or bad: it follows the same distribution we imposed for the single-project firm in the baseline model. Moreover, we assume that the liquidity shocks of both projects are perfectly correlated. [18]

Moreover, since the model now involves two projects, we need to adjust the assumptions properly. In what follows we will keep the Assumptions 1 through 3 but add the Assumption 5 and replace the Assumption 4 with

Assumption 6.

$$\underline{w}^{2proj} \leq w < 2I + 2\rho \tag{20}$$

where

$$\underline{w}^{2proj} := 2I+(1+q)\lambda\rho-[(1-q)(1-\lambda)(p_B+p_G-2p_Bp_G)-2qp_G(1-p_G)]R-2R\left[\frac{(1-q)(1-\lambda)}{1-\tilde{p}_B\tilde{p}_G}+\frac{q}{1-\tilde{p}_G^2}\right] \tag{21}$$

[17]We impose this assumption to obtain the sharp contrast that the large firm can be *fully* insured and achieves the first best. Without this assumption, the liquidity insurance can be full, but the first best will not be achieved: the firm will continue a bad shocked project when the other is not shocked, and the liquidity insurance exceeds the liquidity cost of one project. Of course, it then might be in the firm's interest to obtain partial insurance as in the single-project case.

[18]We make this assumption simply to reduce the number of cases that we need to discuss. It does not undermine our result as we will obtain full insurance under the Assumptions 5. Therefore, the first best is achieved even if the shocks are independent or arbitrarily correlated.

We also require that

Assumption 7. *The debt contract only leaves a payoff to the entrepreneur when all projects succeed, i.e.,*

$$\frac{2b}{R} \leq p_G^2 - (p_G - \Delta p)^2 \tag{22}$$

In addition, since the insurance is full, the firm's credit line doesn't have cash covenants. So cash diversion before drawing down the credit line is irrelevant. But we still assume that the firm cannot divert the cash *after* drawing down the credit line.

In what follows, let's denote

$$\theta_1 := \theta(p_G) = 1 - \frac{b/R}{\Delta p}; \text{ and } \theta_2 := \theta(p_G, p_G) = 1 - \frac{b/R}{p_G^2 - (p_G - \Delta p)^2} \tag{23}$$

for the pledgeability when the firm has one and two good projects, respectively.

The following lemma characterizes the first best

Lemma 8. *The first best is achieved if*

1. *Both projects are funded;*
2. *When the liquidity shocks do not occur, the firm continues both projects and exerts effort;*
3. *When the liquidity shocks occur, the firm always continues Project A and exerts effort, but continues Project B and exerts efforts if and only if it's good.*

The proof is evident.

Proposition 6. *The following strategy profile constitutes a PBE that achieves the first best:*

- *At date $t = 0$, the entrepreneur with two projects funds both projects by borrowing $2I - \underline{w}^{2proj}$ from the creditor and promise to repay $D = 2\theta_2 R$, chips in $\underline{w}^{2proj}$, and obtain a credit line $l = 2\rho$ from the same lender.*
- *At date $t = 1$, if both projects are not shocked, the firm continues both projects and exerts efforts on both projects.*
- *At date $t = 1$, if both projects are shocked and the second project is good, the firm drawdowns the credit line $\ell = 2\rho$, continues both projects, and exerts efforts on both projects.*
- *If instead Project B is bad, the firm draws down half of the credit line and only continues Project A (the good project). He then renegotiates the debt down to $D' = \theta(p_G)R$ and exerts effort.*

- *At date 2, if both projects are continued, the firm repays $2\theta_2 R$ to the lender and $\theta_1 R$ if only one is continued.*

Note here the lender's belief no longer matters because the credit line, in this case, is unconditional, and the lender cannot revoke it no matter what happens.

A big difference between this large-firm case and the previous small-firm case in the baseline model is that large firms are better insured: i) they have a relatively higher credit limit with respect to their size; ii) they don't have covenants in their credit lines; iii) they have a lower credit spread. These facts are consistent with the empirical findings. Greenwald et al. (2021) shows that small firms rely more on term loans than credit lines and have a lower unused-to-used credit ratio. They also report that large firms have few covenants violations and hence unrestricted credit access. Davydenko et al. (2020) also find that cov-lite loans are concentrated among large firms. Chodorow-Reich et al. (2022) showed that the credit spreads distribution of small firms first-order stochastically dominates that of large.

7 Extensions and discussions

7.1 Continuous types

In the baseline model, we only consider the binary types. Now we extend it to a more general setting where the type p follows a continuous distribution F with full support on $[0, 1]$. The following lemma now characterizes the first best.

Lemma 9. *The first best is achieved if*

1. *the project is funded if and only if*

$$\lambda \int_{p^*}^{1} (pR - \rho)\mathrm{d}F(p) + (1 - \lambda) \int_{0}^{1} pR\mathrm{d}F(p) \geq I \tag{24}$$

2. *the project is always continued when there's no liquidity shock; when there's a liquidity shock, i.e., $\tilde{\rho} = \rho$, the project is continued if and only if $p \geq p^*$*

where

$$p^* := \frac{\rho}{R}. \tag{25}$$

This lemma says the first best is achieved if both ex-post and ex-ante efficiency are achieved. This lemma shows that the first best can be achieved with a credit line with a cash covenant in the following proposition.

Proposition 7. *The following action profiles and beliefs consist of a PBE, and the first best is achieved in the PBE:*

- *At date $t = 0$, the entrepreneur borrows $I + p^*(1-\theta)R - \underline{w}^{Cont.Type}$ with a debt of face value $D_0 = \theta R$ and chips in his own capital $\underline{w}^{Cont.Type}$ where*

$$\underline{w}^{Cont.Type} = I + \lambda\rho F(p^*) - \left[\lambda\theta R \int_{p^*}^{1} p\mathrm{d}F(p) + (1-\lambda)\theta \int_{0}^{1} p\mathrm{d}F(p)\right] R. \tag{26}$$

  In addition, he obtains a credit line $l = \rho - (1-\theta)p^*R$ from a lender with the cash covenant that he could draw down the credit line only if he maintains a cash reserve of $(1-\theta)p_B R$. The entrepreneur keeps a cash reserve $c = (1-\theta)p^*R$

- *At date $t = 1$, if the liquidity shock does not occur $\tilde{\rho} = 0$, the entrepreneur pays a cash dividend $c = p^*(1-\theta)R$ to himself*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is above the threshold, i.e., $p \geq p^*$, the entrepreneur does not divert cash and draws down the line of credit, and he uses the funds to continue the project.*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is below the threshold, i.e., $p < p^*$, the entrepreneur diverts cash and the lender revokes the credit line. The project is liquidated.*

The lender believes the quality of the project follows a distribution with density $\frac{f(p)\mathbb{1}_{p \geq p^*}}{1-F(p^*)}$ if the entrepreneur does not divert the cash; and a distribution with density $\frac{f(p)\mathbb{1}_{p<p^*}}{F(p^*)}$ if the entrepreneur diverts the cash.

7.2 Application in Banking

Despite that we call the borrower "firm" in the model, nothing prevents us from thinking of it as a bank. Indeed, it harbors the key element we see in a bank: maturity mismatch. It holds a long-term asset but faces short-term liquidity shock, which could be a sporadic run on the bank. And the maintenance cost ρ is duly interpreted as the demand deposit. The quality of the asset is mapped to the bank's loan portfolio, where a bank is of low quality if it has a lot of non-performing loans. The credit line in the model could be interpreted as the Discount Window borrowing from the Fed or the Committed Liquidity Facility (CLF) used by the Reserve Bank of Australia. (See Copeland et al. (2021) for more details). As documented by Armantier et al. (2015), more than 99% of Discount Window borrowing comes from financially strong and well-capitalized banks through the primary credit program.

8 Conclusion

In this paper, we explore the relationship between liquidity insurance and pledgeability. The use of liquidity insurance was motivated by the lack of pledgeability to raise liquidity when the shock occurs and, according to classical theory, should be unconditional and irrevocable. The paper tries to reconcile the liquidity-insurance view with the paradoxical observations: credit lines usually have covenants whose violation typically leads to the revocation of the credit access; the access to cash often requires keeping a good amount of cash; firms that need the liquidity insurance the most are often insured the least and revoked the most. We add one element to the classic theory that the quality of the project is only known to the firm and is uncontractable, which speaks to a typical drawback of full insurance: firms tend to continue the projects even when it's socially inefficient to continue as the liquidity insurance is borne entirely by the lender. A good design of liquidity insurance should require the firm to put some contingent skin in the game so the continuation decision is aligned. The extent of the liquidity insurance depends crucially on the cash flow pledgeability: Firms with low pledgeability must put more cash into co-investment, as they typically benefit more from the continuation of an efficient project. Thus, in contrast to the classical view that low pledgeability motivates the use of liquidity insurance, we also find low pledgeability limits the extent of liquidity insurance.

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A Proofs for Section 4 (Benchmarks and Main Results)

Lemma 1. *If $w \geq I + \rho$, the following action profile constitutes a PBE:*

1. *The entrepreneur incurs initial cost I to the fund the project at $t = 0$.*
2. *At $t = 1$, if the project is shocked ($\tilde{\rho} = \rho$) and if the quality is good ($Q = G$), the entrepreneur pays the liquidity cost ρ and continues the project; if the project is not shocked, the entrepreneur always continues the project no matter the quality.*

And the first best is achieved in this equilibrium.

Proof. It is straightforward to see that the actions are efficient ex post from the Definition 1. It is sufficient to show these actions are feasible and sequentially rational. To implement this action profile, the firm would need to incur the initial cost I and keep a cash reserve of ρ in preparation for the liquidity shock. This is feasible given $w \geq I + \rho$.

To see why this is sequentially rational, let's stand at $t = 1$ and see what's the best action the entrepreneur can take: if the project is not shocked, liquidating it only gives 0 while continuing it yields at least $p_G R$, so the entrepreneur continues no matter the type. If the project is shocked, continuing the project yields $p_Q R - \rho$, which is positive if and only if $Q = G$ by Assumption 1. So the entrepreneur continues if and only if the project is good. Knowing that the project will be liquidated efficiently, the entrepreneur funds the project at date 0 since it has a positive NPV per Assumption 2. This concludes the proof. □

Lemma 2. *The firm cannot achieve the first best by borrowing in the spot credit market at date* 1 *when the liquidity shock occurs.*

Proof. We first consider senior debt issuance at date 1 and show that the first best cannot be achieved. We prove it by contradiction. Suppose there exists an equilibrium that achieves the first best, in which the firm borrows $I - w$ with a junior debt of face value D_0 at date 0 to fund the project and borrows at least ρ with a senior debt of face value D_1 from *another* creditor at date 1 when the liquidity shock occurs. Since the maximum payoff of the project when the project is good is $\theta p_G R$ which is less than the liquidity cost ρ by Assumption 3, an external creditor would not be willing to lend even if he can fully dilute the existing creditor. *A fortiori*, a debt of lower priority, i.e., pari passu or subordinated, gives the external creditor an even lower payoff, and borrowing is also not feasible.

Now let's consider the case of borrowing from the existing creditor. The existing creditor may be willing to lend without breaking even on the new debt D_1 if the recovery from the initial debt D_0 is enough to compensate for the loss on the new debt. Suppose R_1 is repaid to the new debt while R_0 is repaid to the initial debt D_0, the loss from the new debt, when the state is good, is

$$\rho - p_G R_1 \tag{27}$$

To compensate for the loss in the new loan, he must recover at least this much from the initial debt, i.e.,

$$p_G R_0 \geq \rho - p_G R_1. \tag{28}$$

But this is not possible given the limited pledgeability assumption (Assumption 3) that $R_0 + R_1 \leq \theta R$. Therefore, borrowing from the existing is not feasible as well. This concludes our proof. □

Lemma 3. *The firm cannot achieve the first best with cash hoarding $c = \rho$ if either one of the following conditions is met*

1. $\rho > (1-\theta)p_G R$;
2. $D_0 < R - \frac{\rho}{p_B}$ *if condition 1 is not met;*
3. $w < \underline{w}^{Cash} := I + \rho - [q p_G + (1-q)(1-\lambda)p_B]\theta R$ *if both conditions 1 and 2 are not met.*

Proof. We prove the result by contradiction. Suppose the firm borrows D_0 to fund the project, keeps a cash reserve c at date 0, and achieves the first best. In order to mitigate the liquidity shock, the cash reserve needs to be at least $c \geq \rho$. Thus, the firm must borrow at least $I - (w - \rho)$ with a debt of face value D_0 to finance the project. And the firm has an obligation to pay $\min\{I - (w - \rho), \theta R\}$ to the creditor at date 2 when the project matures.

Now let's analyze the continuation incentive to continue. When the liquidity shock is absent, it's sequentially rational for the firm to divert cash. When the shock occurs, the firm expects to get

$$p_Q \max\{(1-\theta)R, R - D_0\} \tag{29}$$

so he diverts cash if and only if $p_Q \max\{(1-\theta)R, R - D_0\} < c$. To achieve the first best, it must be that

$$p_B \max\{(1-\theta)R, R - D_0\} \leq \rho \leq p_G \max\{(1-\theta)R, R - D_0\} \tag{30}$$

When $D_0 \geq \theta R$, the inequality holds if $p_B(1-\theta)R < \rho \leq p_G(1-\theta)R$; When $D_0 < \theta R$, the condition holds if and only if

$$R - \frac{\rho}{p_B} < D_0 \leq R - \frac{\rho}{p_G} \tag{31}$$

The first inequality implies $(1-\theta)p_B R < \rho$, which is always satisfied given our Assumption 1, while the second is satisfied when $\rho \leq (1-\theta)p_G R$. So the condition could be violated if either i) $\rho > p_G(1-\theta)R$, or ii) $\rho \leq p_G(1-\theta)R$ but $D_0 < R - \frac{\rho}{p_B}$.

Even if the above conditions are met, in order for the creditor to break even, it must be that

$$I - (w - \rho) \leq [qp_G + (1-q)(1-\lambda)p_B] \min\{\theta R, D_0\} \tag{32}$$

and the binding constraint is

$$w \geq \underline{w}^{Cash} = I + \rho - [qp_G + (1-q)(1-\lambda)p_B]\theta R. \tag{33}$$

This completes the proof of the lemma.

□

Proposition 1. *There exists some w for which cash hoarding cannot achieve first best, but a credit line with cash covenants can.*

Proof. To prove the result, let's notice that even when $\rho \leq (1-\theta)p_G R$ and $D_0 \geq R - \frac{\rho}{p_G}$, we have that

$$\underline{w}^{Cash} - \underline{w}^{Cov} = (1 - \lambda q)(\rho - (1-\theta)p_B R) > 0. \tag{34}$$

Therefore, for any $w \in (\underline{w}^{Cov}, \underline{w}^{Cash})$, the first best is not achieved by cash hoarding but by using a credit line with cash covenants. Thus regardless of the size of the shock, there's always some value of w for which cash hoarding cannot achieve first best but a credit line with cash covenants. (See the proof of Proposition 4 for the latter.) □

Proposition 2. *Suppose the entrepreneur obtains a line of credit $\ell = \rho$ from the creditor. In no equilibrium is the first best obtained.*

Furthermore, if $[qp_G + (1-q)p_B]R < I + \lambda\rho$ or $w < \underline{w}^{CL} := I + \lambda\rho - [qp_G + (1-q)p_B]\theta R$, the project is not funded; otherwise, it can be funded with any debt of face value $D_0 \in \left[\frac{I - w + \lambda\rho}{qp_G + (1-q)p_B}, (1-\theta)R\right]$ but is always continued even when it is bad.

Proof. To show that the first best cannot be achieved, it suffices to show that at $t = 1$ when the liquidity shock occurs, the entrepreneur always continues the project. To see this, when the credit line decides whether to continue a bad project or not, he would divert any cash reserves c and then draw down the credit line to continue the bad projects since liquidation only gives him at most c while continuation would give the firm $p_B(1-\theta)R + c$ in expectation. Therefore, in any equilibrium, the first best is not achieved.

Now we continue to show when the project would be funded. Since the lenders break even ex ante, the project is only funded when i) the inefficiently continued project has a positive NPV; ii) the firm can raise enough capital ex ante. Condition i) requires

$$[qp_G + (1-q)p_B]R \geq I + \lambda\rho \tag{35}$$

and condition ii) requires

$$[qp_G + (1-q)p_B]\min\{\theta R, D_0\} \geq I - w + \lambda\rho. \tag{36}$$

And the minimum feasible $\underline{D}_0 = \frac{I-w+\lambda\rho}{qp_G+(1-q)p_B}$ exists only if

$$w \geq \underline{w}^{CL} = I + \lambda\rho - [qp_G + (1-q)p_B]\theta R \tag{37}$$

$$= I + \lambda\rho - (1-q)\lambda p_B \theta R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{38}$$

Therefore, funding is only feasible when the entrepreneur's wealth exceeds the threshold $\underline{w}^{CL}$. And the entrepreneur wants to fund the project only if the expected return $p_q R - (I + \lambda\rho)$ is positive.

This concludes our proof. □

Proposition 3. *If the lender could credibly bribe the firm not to draw down the credit line, the project can be funded with an unconditional credit line.*

Proof. To continue a bad project involves a loss of $-(p_B \min\{\theta R, D_0\} - \rho)$ while bribing to terminate involve a loss of at least $p_B \max\{(1-\theta)R, R - D_0\}$, so the lender is willing to bribe if

$$p_B \min\{\theta R, D_0\} - \rho \leq -p_B \max\{(1-\theta)R, R - D_0\} \tag{39}$$

which can be rearranged to $p_B R \leq \rho$, i.e., our assumption 1; iii) the firm can raise enough capital. Now the lender's break-even condition is

$$[qp_G + (1-q)(1-\lambda)p_B]\min\{\theta R, D_0\} = I - w + q\lambda\rho + (1-q)\lambda p_B \max\{(1-\theta)R, R - D_0\} \tag{40}$$

which is equivalent to

$$w = I + q\lambda\rho - [qp_G + (1-q)p_B]\min\{\theta R, D_0\} + (1-q)\lambda p_B R. \tag{41}$$

If a D_0 exists such that the condition is satisfied, then funding and the first best is feasible.

And the relevant binding constraint is

$$w \geq \underline{w}^{Renegotiation} = I + q\lambda\rho + (1-q)\lambda p_B R - [qp_G + (1-q)p_B]\theta R \tag{42}$$

$$= I + q\lambda\rho + (1-q)\lambda p_B(1-\theta)R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{43}$$

Compare this threshold with $\underline{w}^{Cov}$, we have

$$\underline{w}^{Cov} - \underline{w}^{Renegotiation} = (1-\lambda)p_B(1-\theta)R > 0 \tag{44}$$

Since this threshold $\underline{w}^{Renegotiation}$ is lower than $\underline{w}^{Cov}$, the project can always be funded.

□

Proposition 4. *The following action profiles and beliefs consist of a PBE, and the first best is achieved in the PBE:*

- *At date $t = 0$, the entrepreneur borrows $I + p_B(1-\theta)R - \underline{w}^{Cov}$ with a debt of face value $D_0 = \theta R$ and chips in his own capital $\underline{w}^{Cov}$ where*

$$\underline{w}^{Cov} = I + q\lambda\rho + (1-q\lambda)p_B(1-\theta)R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{10}$$

  as defined in Assumption 4. In addition, he obtains a credit line $l = \rho - (1-\theta)p_B R$ from a lender with the cash covenant that he could draw down the credit line only if he maintains a cash reserve of $(1-\theta)p_B R$. The entrepreneur keeps a cash reserve $c = (1-\theta)p_B R$

- *At date $t = 1$, if the liquidity shock does not occur $\tilde{\rho} = 0$, the entrepreneur pays a cash dividend $c = p_B(1-\theta)R$ to himself*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is good $Q = G$, the entrepreneur does not divert cash and draws down the line of credit, and he uses the funds to continue the project.*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is bad $Q = B$, the entrepreneur diverts cash and the lender revokes the credit line. The project is liquidated.*

The lender believes that the project is good with probability 1 if and only if the entrepreneur does not divert the cash.

Proof. Since the project is always funded and is only continued when it's good, it is obvious that the first best is achieved. Now we need to establish that the strategies are feasible and sequentially rational. We use backward induction, starting from the second period.

Given a debt $D_0 = \theta R$ and the pledgeability condition in Assumption 3, if the liquidity shock occurs, the firm can get c if he chooses to divert cash and not draw down the credit line. (As we discussed, the firm cannot draw down the credit line and divert cash.) Otherwise, continuing the project would yield an expected payoff of $(1-\theta)p_Q R$ so the firm only continues the good project given $c = (1-\theta)p_B R$. (The firm is indifferent between continuing a bad project or diverting cash and terminating the project. Since terminating the bad project is socially efficient, we assume that's the firm's choice too. Indeed, the firm can credibly commit to doing so, or he won't be able to get financed in the first place. Another way to reconcile this problem is to change the cash covenant to $c = (1-\theta)p_B R + \varepsilon$ for an arbitrarily small ε. This is always an equilibrium as long as the wealth constraint is not binding.)

Now let's examine the feasibility at date 0. The firm needs to borrow at least $I + p_B(1-\theta)R - w$ to start the project at date 0 and an additional drawdown of $\rho - (1-\theta)p_B R$ with probability $q\lambda$. The lender is repaid $D_0 = \theta R$ whenever there's no liquidity shock, and the project succeeds, with probability $(1-\lambda)(qp_G + (1-q)p_B)$, or when there's a liquidity shock, and the project is good and succeeds, with probability $\lambda q p_G$. The break-even condition for the lender to enter the contract is

$$I + p_B(1-\theta)R - w + q\lambda(\rho - p_B(1-\theta)R) \leq [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{45}$$

Rewriting,

$$w \geq \underline{w}^{Cov} = I + q\lambda\rho + (1-q\lambda)p_B(1-\theta)R - [qp_G + (1-q)(1-\lambda)p_B]\theta R \tag{46}$$

which holds by Assumption 4. Therefore, the first best is achieved in with a credit line with cash covenants. □

Proof. We simply take derivative of $\ell = \rho - (1-\theta)p_B R$ with respect to θ and get

$$\frac{\mathrm{d}\ell}{\mathrm{d}\theta} = p_B R > 0. \tag{47}$$

Done. □

Proposition 5. *A credit line with reduced credit limit $l = \rho - p_B(1-\theta)R$ but no cash covenants cannot achieve the first best.*

Proof. The idea is to show that the firm can partially liquidate the inefficient project and continue the downsized inefficient project. This is more efficient than full insurance but is still not the first best.

So suppose the firm also keeps a cash reserve $p_B(1-\theta)R$ and the credit line $\rho - p_B(1-\theta)R$. When the liquidity shock occurs and the project is bad, the firm can either divert the cash, draw down the limited credit line, and continue the reduced-size project, or use its cash as co-investment and anything in between. Note that the firm would divert any cash not used as co-investment as keeping them on balance would only benefit the creditors.

Let the x be the fraction of the project that is not liquidated and continued. If $x \le \ell/\rho$, the credit line can continue the entire x fraction of the project, and the firm can pocket in all the cash reserves. If instead, the $x > \ell/\rho$, the firm has to chip in $x\rho - \ell > 0$ so the firm can only divert $c - (x\rho - \ell)$ And continuing this fraction of the project would give the firm an ex-interim payoff of

$$U(x) := p_B \max\{x(1-\theta)R, xR - D_0\} + c - \max\{x\rho - l, 0\} \tag{48}$$

Taking derivative w.r.t. x, we have

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} = \begin{cases} p_B(1-\theta)R + p_B\theta R \mathbb{1}_{x > \frac{D_0}{\theta R}} > 0 & \text{if } x \le \frac{\ell}{\rho} \\ p_B(1-\theta)R + p_B\theta R \mathbb{1}_{x > \frac{D_0}{\theta R}} - \rho < 0 & \text{if } x > \frac{\ell}{\rho} \end{cases} \tag{49}$$

therefore the function $U(x)$ has a global maximum at $x = \frac{\ell}{\rho}$ and thus, the optimal action for the firm is to liquidate a fraction $\frac{p_B(1-\theta)R}{\rho}$, divert the cash and draw down the credit lines to continue the project. Thus, only a capped credit line without cash covenants would not implement the first best.

The reason is that when $x < \ell/\rho$, the firm can divert no more cash than $p_B(1-\theta)R$ but liquidation of more of the project only reduces his payoff from continuation; however, when $x > \frac{\ell}{\rho}$, it's the firm who's paying the entire marginal cost of continuing the additional fraction of the project, and therefore, he's not willing to continue an additional fraction of the project.

This concludes our proof.

□

B Proofs for Section 5 (Pledgeability with Multiple Projects)

Lemma 4. *Consider two project profiles $\boldsymbol{p}^\alpha = (p_1, \ldots, p_j^\alpha, \ldots, p_n)^\top$ and $\boldsymbol{p}^\beta = (p_1, \ldots, p_j^\beta, \ldots, p_n)^\top$ that only differs in element j, and $p_j^\alpha > p_j^\beta$, we have $\theta(\boldsymbol{p}^\alpha) > \theta(\boldsymbol{p}^\beta)$.*

Proof. To show that project profile $\boldsymbol{p}^{\alpha}$ admits a higher pledgeability, it suffices to show

$$p_j^{\alpha}\prod_{j\neq i} p_i - (p_j^{\alpha}-\Delta p)\prod_{j\neq i}(p_i-\Delta p) \geq p_j^{\beta}\prod_{j\neq i} p_i - (p_j^{\beta}-\Delta p)\prod_{j\neq i}(p_i-\Delta p) \tag{50}$$

Rearranging

$$(p_j^{\alpha}-p_j^{\beta})\left[\prod_{j\neq i} p_i - \prod_{j\neq i}(p_i-\Delta p)\right] \geq 0 \tag{51}$$

which is always true since $p_j^{\alpha} > p_j^{\beta}$ by assumption and $\prod_{j\neq i} p_i > \prod_{j\neq i}(p_i-\Delta p)$. ☐

Lemma 5. *Consider two project profiles $\boldsymbol{p}^{(n)} = (p_1, ..., p_n)^{\top}$ and $\boldsymbol{p}^{(n+1)} = (p_1, ..., p_{n+1})^{\top}$, $\theta(\boldsymbol{p}^{(n)}) \leq \theta(\boldsymbol{p}^{(n+1)})$ if and only if*

$$p_{n+1} \geq 1 - \left(\prod_{i=1}^{n} \frac{p_j}{p_j-\Delta p} - 1\right)^{-1}\Delta p \tag{14}$$

Proof. Adding one project increases the pledgeability, i.e., $\theta(\boldsymbol{p}^{(n)}) \leq \theta(\boldsymbol{p}^{(n+1)})$ if and only if

$$\prod_{i=1}^{n} p_i - \prod_{i=1}^{n}(p_i-\Delta p) \leq \prod_{i=1}^{n+1} p_i - \prod_{i=1}^{n+1}(p_i-\Delta p) \tag{52}$$

Rearranging

$$(1-p_{n+1})\prod_{i=1}^{n} p_i \leq (1+\Delta p - p_i)\prod_{i=1}^{n}(p_i-\Delta p) \tag{53}$$

$$p_{n+1} \geq 1 - \left(\prod_{i=1}^{n} \frac{p_j}{p_j-\Delta p} - 1\right)^{-1}\Delta p \tag{54}$$

☐

Lemma 6. *Consider two project profiles $\boldsymbol{p}^{\alpha} = (p_1, ..., p_j^{\alpha}, ..., p_n)^{\top}$ and $\boldsymbol{p}^{\beta} = (p_1, ..., p_j^{\beta}, ..., p_n)^{\top}$ that only differs in element j, and $p_j^{\alpha} > p_j^{\beta}$, we have $\bar{R}_b(\boldsymbol{p}^{\alpha}) > \bar{R}_b(\boldsymbol{p}^{\beta})$.*

Proof. Notice we can write $\bar{R}_b(\boldsymbol{p})$ as

$$\bar{R}_b(\boldsymbol{p}) = \frac{nb}{1-\tilde{p}_j\prod_{i\neq j}\tilde{p}_j} \tag{55}$$

which is increasing in $\tilde{p}_j$. So $p_j^{\alpha} > p_j^{\beta}$ implies $\tilde{p}_j^{\alpha} > \tilde{p}_j^{\beta}$, which in turn implies $\bar{R}_b(\boldsymbol{p}^{\alpha}) > \bar{R}_b(\boldsymbol{p}^{\beta})$ by the monotonicity of $\bar{R}_b(\cdot)$ in each element. ☐

Lemma 7. *Consider two project profiles $\boldsymbol{p}^{(n)} = (p_1, ..., p_n)^\top$ and $\boldsymbol{p}^{(n+1)} = (p_1, ..., p_{n+1})^\top$, $\bar{R}_b(\boldsymbol{p}^{(n+1)}) \geq \bar{R}_b(\boldsymbol{p}^{(n)})$ if and only if $\tilde{p}_{n+1} \geq 1 - \frac{\prod_{j=1}^n \tilde{p}_j^{-1} - 1}{n}$, or equivalently*

$$p_{n+1} \geq \left(\prod_{j=1}^{n} \frac{p_j}{p_j - \Delta p} - 1 \right)^{-1} n\Delta p \tag{17}$$

Proof. Using the definition of $\bar{R}_b$ in Equation (15), we have that $\bar{R}_b(\boldsymbol{p}^{(n+1)}) \geq \bar{R}_b(\boldsymbol{p}^{(n)})$ is equivalent to

$$\frac{(n+1)b}{1 - \tilde{p}_{n+1} \prod_{j=1}^n \tilde{p}_j} \geq \frac{nb}{1 - \prod_{j=1}^n \tilde{p}_j}. \tag{56}$$

Multiplying the denominators on both sides and rearranging, we have

$$b \geq (n+1)b \prod_{j=1}^{n} \tilde{p}_j - nb\tilde{p}_{n+1} \prod_{j=1}^{n} \tilde{p}_j \tag{57}$$

Rearranging

$$\tilde{p}_{n+1} \geq 1 - \frac{\prod_{j=1}^n \tilde{p}_j^{-1} - 1}{n} \tag{58}$$

Using the definition of $\tilde{p}_{n+1}$

$$1 \geq \left(1 + \frac{n\Delta p}{p_{n+1}}\right) \prod_{j=1}^{n} \tilde{p}_j \tag{59}$$

Or equivalently

$$p_{n+1} \geq \left(\prod_{j=1}^{n} \tilde{p}_j^{-1} - 1 \right)^{-1} n\Delta p \tag{60}$$

Substituting $\tilde{p}_j$ gives the expression in the lemma, concluding what we want to prove. □

C Proofs for Section 6 (Firm Size and Access to Liquidity Insurance)

Proposition 6. *The following strategy profile constitutes a PBE that achieves the first best:*

- *At date $t = 0$, the entrepreneur with two projects funds both projects by borrowing $2I - \underline{w}^{2proj}$ from the creditor and promise to repay $D = 2\theta_2 R$, chips in $\underline{w}^{2proj}$, and obtain a credit line*

$l = 2\rho$ from the same lender.

- *At date $t = 1$, if both projects are not shocked, the firm continues both projects and exerts efforts on both projects.*
- *At date $t = 1$, if both projects are shocked and the second project is good, the firm drawdowns the credit line $\ell = 2\rho$, continues both projects, and exerts efforts on both projects.*
- *If instead Project B is bad, the firm draws down half of the credit line and only continues Project A (the good project). He then renegotiates the debt down to $D' = \theta(p_G)R$ and exerts effort.*
- *At date 2, if both projects are continued, the firm repays $2\theta_2 R$ to the lender and $\theta_1 R$ if only one is continued.*

Proof. It's obvious from the Lemma 8 that the action profile implements first best. Now we want to prove it's feasible and incentive compatible. We analyze it using backward induction.

At date $t = 1$, when the liquidity projects do not occur, the firm cannot do anything but continue the projects. When the liquidity shock occurs and both projects are good, continuing both projects yields in expectation $p_G p_G 2(1-\theta_2)R$ while continuing only one project yields $p_G(1-\theta_1)R$, we have

$$p_G p_G 2(1-\theta_2)R \geq p_G(1-\theta_1)R \tag{61}$$

which always holds. So the firm is willing to continue both projects.

However, if Project B turns out to be bad, the payoff from continuing both projects is reduced to $p_G p_B 2(1-\theta(p_G, p_B))R$, and we have

$$p_G p_B 2(1-\theta(p_G, p_B))R = \frac{2p_G p_B b/R}{p_G p_B - (p_G - \Delta p)(p_B - \Delta p)} \leq p_G(1-\theta_1)R \tag{62}$$

by Assumption 5. Therefore, the firm only draws down half of the credit line and only continues Project A.

Now we proceed to check the feasibility of the lending. We break down the payoff of the lender

- If Project B is good and the liquidity shock doesn't occur, the lender gets $2\theta_2 R$ with both projects succeed, which occurs with conditional probability $p_G p_G$; gets R when only one project succeeds, which occurs with probability $2p_G(1-p_G)$;

- If Project B is good and the liquidity shock occurs, the lender gets $2\theta_2 R - 2\rho$ with both projects succeed, which occurs with conditional probability $p_G p_G$; gets $R - 2\rho$ when only one project succeeds, which occurs with probability $2p_G(1-p_G)$, and -2ρ when no projects succeed;

- If Project B is bad and the liquidity shock doesn't occur, the lender gets $2\theta(p_G, p_B)R$ with both projects succeed, which occurs with conditional probability $p_G p_B$; gets R when only one project succeeds, which occurs with probability $p_G(1-p_B)+p_B(1-p_G)$;

- If Project B is good and the liquidity shock occurs, the lender gets $\theta_1 R - \rho$ if the continued Project A succeeds, which occurs with probability p_G

So the break-even condition of the firm is

$$(1-q)((1-\lambda)(2p_G p_B \theta(p_G, p_B)R + (p_G(1-p_B) + p_B(1-p_G)R)) + \lambda(p_G\theta_1 R - \rho) \tag{63}$$

$$q(2p_G p_G \theta_2 R + 2p_G(1-p_G)R - 2\lambda\rho) \geq 2I - w \tag{64}$$

which can be simplified to

$$w \geq 2I+(1+q)\lambda\rho-[(1-q)(1-\lambda)(p_B+p_G-2p_B p_G)-2qp_G(1-p_G)]R-2R\left[\frac{(1-q)(1-\lambda)}{1-\tilde{p}_B\tilde{p}_G}+\frac{q}{1-\tilde{p}_G^2}\right] \tag{65}$$

which holds due to our Assumption 6. So the project can be founded ex ante □

D Proofs for Section 7 (Extensions and discussions)

Proposition 7. *The following action profiles and beliefs consist of a PBE, and the first best is achieved in the PBE:*

- *At date $t=0$, the entrepreneur borrows $I + p^*(1-\theta)R - \underline{w}^{Cont.Type}$ with a debt of face value $D_0 = \theta R$ and chips in his own capital $\underline{w}^{Cont.Type}$ where*

$$\underline{w}^{Cont.Type} = I + \lambda\rho F(p^*) - \left[\lambda\theta R\int_{p^*}^{1} p\mathrm{d}F(p) + (1-\lambda)\theta\int_{0}^{1} p\mathrm{d}F(p)\right]R. \tag{26}$$

In addition, he obtains a credit line $l = \rho - (1-\theta)p^*R$ from a lender with the cash covenant that he could draw down the credit line only if he maintains a cash reserve of $(1-\theta)p_B R$. The entrepreneur keeps a cash reserve $c = (1-\theta)p^*R$

- *At date $t = 1$, if the liquidity shock does not occur $\tilde{\rho} = 0$, the entrepreneur pays a cash dividend $c = p^*(1-\theta)R$ to himself*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is above the threshold, i.e., $p \geq p^*$, the entrepreneur does not divert cash and draws down the line of credit, and he uses the funds to continue the project.*

- *At date $t = 1$, if the project is shocked $\tilde{\rho} = \rho$ and the quality is below the threshold, i.e., $p < p^*$, the entrepreneur diverts cash and the lender revokes the credit line. The project is liquidated.*

The lender believes the quality of the project follows a distribution with density $\frac{f(p)\mathbb{1}_{p\geq p^*}}{1-F(p^*)}$ if the entrepreneur does not divert the cash; and a distribution with density $\frac{f(p)\mathbb{1}_{p<p^*}}{F(p^*)}$ if the entrepreneur diverts the cash.

Proof. First, notice that the ex-post efficiency is obvious, and the posterior belief is calculated using the Bayes' Rule, so we only need to check that the continuation decision is incentive compatible and funding is feasible. With cash covenants, the firm never wants to downsize. And he wants to continue a project if and only if

$$p(1-\theta)R \geq c \implies p \geq p^*. \tag{66}$$

So the first best threshold is incentive compatible.

Moreover, the break-even condition of the lender is

$$\lambda \int_{p^*}^{1} (p\theta R - \rho)\mathrm{d}F(p) + (1-\lambda)\theta \int_{0}^{1} pR\mathrm{d}F(p) \geq I - w \tag{67}$$

which is equivalent to

$$w \geq \underline{w}^{Cont.Type} = I + \lambda\rho F(p^*) - \left[\lambda\theta R \int_{p^*}^{1} p\mathrm{d}F(p) + (1-\lambda)\theta \int_{0}^{1} p\mathrm{d}F(p)\right] R \tag{68}$$

which is feasible, given our assumption. □