SYSTEMIC RISK IN FINANCIAL NETWORKS REVISITED: THE ROLE OF MATURITY*

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Abstract

We develop a model of interbank networks in which banks experience state-contingent liquidity shocks. We show that networks of long-term debt facilitate the efficient transfer of liquidity: They allow shocked banks to raise liquidity using interbank claims as collateral for new debt, diluting interbank liabilities. Networks of long-term debt thus have strikingly different properties from those of short-term debt, which cannot be diluted; e.g., high indebtedness and connectedness can be sources of stability, not fragility. Networks in a specific class, which we call "exponential networks," implement optimal contingent transfers despite consisting of plain (non-contingent) debt—they are robust but never fragile.

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1 Introduction

Banks are connected in networks of debt. Unlike in the Walrasian model, in which only net positions matter, gross positions are thought to be a source of systemic risk. A number of theory papers support this conclusion, showing, inter alia, that tightly inter-connected network structures are "robust yet fragile," absorbing everyday shocks but amplifying extraordinary ones (see, notably, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), hereinafter AOT, and Allen and Gale (2000)). Banks maintain these positions even though practitioners and policy makers alike champion netting them out, saying, e.g., that "Support for netting is well-nigh universal in the financial industry as well as among policy makers" (Mengle (2010), p. 2).

The literature has focused on short-term (one-period) debt, capturing, e.g., repo markets. But, in practice, interbank debts often have longer maturity.[1] In this paper, we develop a financial networks model of long-term interbank debt. Do interbank networks of long-term debts harbor the same systemic risks of those of short-? Do the same network structures lead risks to propagate? Do they serve an economic function that could be undermined by netting out debts?

We find that high indebtedness and connectedness can be sources of stability in long-term debt networks, in diametric contrast to short-. Networks in a specific class, which we call the "exponential networks," implement the efficient transfers of liquidity no matter the distribution of shocks. They are robust but never fragile.

One feature of long-term debt underlies our results: It embeds the option to dilute with new debt to a third party. For illustration, consider a bank in the network with interbank debts on both sides of its balance sheet, claims on other banks on the left and liabilities to other banks on the right. If the debts are long-term, it can raise new debt to meet a liquidity

[1]Several papers use data on interbank debt in Germany. They find an average maturity longer than a year and a fraction of overnight debt of about 10% (see, e.g., Bluhm, Georg, and Krahnen (2016), Craig and Ma (2021), Craig and Von Peter (2014), Gabrieli and Georg (2014), and Upper and Worms (2004)). Kuo et al. (2014) point to the scarcity of data on the maturity of US interbank debt and develop a method to impute it from payments data, which suggests that about a quarter of it is term debt.

shock. To do so, it uses its claims on other banks as collateral while diluting its liabilities to other banks. Not so with short-term debt. Being due right away, it cannot be diluted. Thus, whereas all interbank networks can create collateral on the left-hand side of bank balance sheets, only short-term networks encumber that collateral on the right.

To capture banks' liquidity risk, we employ elements of Holmström and Tirole (1998) in a financial networks model. Banks in the model have a maturity mismatch: They have long-term assets but could suffer a liquidity shock in the short term. We assume that their assets are not perfectly pledgeable. This gives liquidity risk bite: Banks could be unable to raise liquidity by pledging their assets in the market, and could be inefficiently liquidated as a result.

We start with the benchmark of short-term debt networks before turning to our main analysis, of long-term debt. We demonstrate that the benchmark model is isomorphic to AOT's, mutatis mutandis. Thus we can apply their measures of connectedness such as delta connectedness, the bottleneck parameter, and the harmonic distance. From their analysis, we know that (i) netting out short-term debts increases financial stability (Lemma 2); (ii) more connected networks (appropriately defined) are less stable for large shocks (Lemma 3 and Lemma 5); and (iii) there is a "default radius" around a negatively shocked bank, i.e. closely connected banks also default (Lemma 4). Behind all these results is the idea that short-term liabilities encumber banks' assets, preventing them from raising liquidity to meet shocks.

We begin our analysis of the long-term debt network by showing the existence and generic uniqueness of a payment equilibrium (Proposition 1). We then establish two sets of main results.

The first contrasts the long-term debt network to the short-. We show that, in diametric contrast to what happens with short-term debt, (i) netting out long-term debts undermines financial stability (Proposition 2); (ii) more tightly connected networks are more stable for large shocks (Proposition 3 via delta connectedness and Proposition 5 via the bottleneck parameter); and (iii) there is a "salvation radius" around a not-shocked bank, i.e. closely connected banks are *not* liquidated (Proposition 4 using the harmonic distance).

Although all long-term debt networks enhance financial stability in our model, many are still inefficient, in that more banks than necessary are liquidated. In particular, some banks suck liquidity out of the system only to be liquidated anyway. For example, any symmetric network allocates excess liquidity equally among distressed banks, when a planner would prioritize them, allocating all the liquidity to the largest subset of banks that it can hope to save and writing off the others entirely (Lemma 6).

Our second set of main results pertains to the exponential networks. In these networks, all banks have debts with all others. But these debts are not the same size. Each bank has larger positions with bank B_i than B_{i+1} for all i, a condition we call "assortativity," which creates an endogenous size distribution. We show that if the positions decay exponentially at a high enough rate, the network is the most efficient no matter the distribution of shocks, in the sense that it allows the greatest number of distressed banks to avoid liquidation (Proposition 6). Intuitively, it prioritizes the allocation of liquidity so that the largest bank always gets the liquidity it needs to survive, the second largest does too as long as there is enough left in aggregate after saving the largest bank, and so on. The network is tightly connected, like the core of real-world interbank networks,[2] and the size distribution is similar to the empirical one, with some banks being "too big to fail" due to only their position in the interbank financial network (as banks are identical in every other way).

Overall, our results provide a new perspective on financial stability. Gross long-term debts can enhance it, suggesting they should not necessarily be netted out—zero-net positions can have positive net present value. Large, highly levered banks can facilitate the allocation of liquidity, suggesting they should not necessarily downsize or recapitalize.

The mechanism in the model reflects practice: All banks maintain gross positions, e.g., cross holdings of loans or bonds, which shocked banks dilute with new senior debt, e.g., repos.

[2] If banks in our model represent the core of a larger network, their liquidity shocks/assets in place could represent their liabilities to/claims on peripheral banks, albeit in reduced form.

It also points to a possible reason why bankers could view the "super senior" bankruptcy treatment of repos as attractive, despite being irrelevant according to the Modigliani–Miller view: It facilitates dilution and therefore efficient contingent transfers.

We analyze two extensions that qualify our rosy view of large long-term debts (Section 6). In one, we allow liquidation to be efficient, so not all banks should meet their shocks; in the other, we assume that any default is costly, not only those that induce inefficient liquidation. In each case, we show how to calibrate debt levels to implement the efficient outcome, albeit only for specific networks. The takeaway is that, per the baseline, debts should be high enough that banks have the unencumbered collateral to weather shocks when liquidation is inefficient, but, now, not so high that they induce too little liquidation or too much default.

In another extension, we allow banks to be heterogeneous, each with different assets in place and/or liquidity shocks. We show that the planner's problem is equivalent to a problem in computer science called the "knapsack problem" (Proposition 8) and that the exponential network implements an algorithm used to solve it called the "greedy algorithm" (Proposition 9). The algorithm need not implement the (constrained) efficient solution. But it often does, e.g., when each bank's liquidity shortfall is the same size (Corollary 2). Moreover, when it does not, the efficiency loss can be bounded, e.g., by the deadweight loss of liquidating one specific bank (Corollary 3). (Such approximate optimality could be the only sensible policy goal, as the knapsack problem is computationally hard.) These results suggest that exponential networks implement a policy that is robust not only to which banks are shocked, but also to their size distribution.

Our paper makes two main contributions to the literature. First, it shows how maturity matters in financial networks, contributing to the networks literature, which is focused on short-maturity debt.[3] Only three other papers study longer maturity debt in networks models, to our knowledge: (i) Allen, Babus, and Carletti (2012), which, unlike us, focuses

[3] Surveys include Allen, Babus, and Carletti (2009), Allen and Walther (2021), Glasserman and Young (2016), and Jackson and Pernoud (2021).

on the maturity of debts to investors outside the network, not among banks within it, (ii) Kusnetsov and Veraart (2018), a mathematical finance paper, which studies the problem of defining and constructing the equilibria in the Eisenberg and Noe (2001) environment with multiple maturities, and (iii) He and Li (2022), which studies how maturity transformation via debt chains can mitigate rollover/resale risk. None of these papers studies debt dilution or the properties of various network structures, our main themes. Second, our paper builds on the idea that default can implement valuable contingencies (notably, Allen and Gale (1998), Dubey, Geanakoplos, and Shubik (1988), and Zame (1993)). We show that the option to dilute provides another layer of contingency on top of the option to default. And we show how network structures can leverage this option: An appropriately constructed network of plain (non-contingent) debt can in fact implement the (constrained) efficient outcome, allocating all available liquidity to the right set of banks, under fairly general conditions.[4] It thus points to an unexplored way that time contingency (maturity) substitutes for state contingency, complementing Angeletos's (2002) idea that the set of bonds of all maturities, whose prices depend on the entire term structure of interest rates, can span all assets (see also Gale (1990)).

The rest of the paper proceeds as follows. Section 2 presents the model. Section 3 considers the short-term debt benchmark. Section 4 states the qualitative properties of long-term debt networks. Section 5 analyzes the exponential network. Section 6 analyzes extensions. Section 7 concludes. The Appendix contains all proofs and a table of notations.

2 Model

We consider a model with two dates $t \in \{1, 2\}$ and $N \geq 2$ agents $\mathrm{B}_1, \mathrm{B}_2, ..., \mathrm{B}_N$, which we refer to as "banks." They resemble real-world banks in so far as each has a maturity mismatch. B_i has assets y in place that pay off at Date 2 and could suffer a liquidity shock

[4] Probably the strongest condition is that, in our model, each bank's liquidation cost is independent of the state, thus so is the planner's order of priority in saving shocked banks. If it were better to save B_i instead of B_j in some states and B_j instead of B_i in others, the exponential network could be constrained inefficient.

ℓ at Date 1.[5] We assume that only the fraction θ of y is valuable to outsiders and the remaining $(1-\theta)y$ accrues to B_i alone, a formulation that serves as a catch-all for numerous agency problems.[6] If B_i cannot meet its liquidity shock its assets are thus liquidated/sold for $\theta y < \ell$.

We write $\sigma_i = 1$ if B_i is shocked and $\sigma_i = 0$ otherwise. There is no other risk, so the "state," which is realized at Date 1, is the profile $\{\sigma_i\}_i =: \boldsymbol{\sigma}$. (We impose no restrictions on its distribution.) There is universal risk neutrality and no discounting.

In addition to their long-term assets and liquidity needs, banks also have debts both to and from other banks—"interbank liabilities" and "interbank claims"—maturing at Date 2. The face value of B_i's liability to B_j is denoted by $F_{i\to j}$, of all its liabilities to other banks by $F_{i\rightrightarrows} := \sum_{j\neq i} F_{i\to j}$, and of its claims on other banks by $F_{i\leftleftarrows} := \sum_{j\neq i} F_{j\to i}$; $\mathbf{F}_{\rightrightarrows}$ denotes the vector with ith element $F_{i\rightrightarrows}$. The matrix $\mathbf{F} := [F_{i\to j}]_{ij}$ defines the interbank network. Following AOT, we assume throughout that it satisfies $F_{i\leftleftarrows} = F_{i\rightrightarrows}$ for all i, so banks have zero net interbank positions. That is a reasonable approximation of reality, as gross interbank positions are often an order of magnitude larger than net.[7]

We denote B_i's equilibrium repayment to B_j by $R_{i\to j}$, its total repayment to all other banks by $R_{i\rightrightarrows} := \sum_{i\neq j} R_{i\to j}$, and its total repayment from all other banks by $R_{i\leftleftarrows} := \sum_{i\neq j} R_{j\to i}$; $\mathbf{R}_{i\rightrightarrows}$ is a vector with ith element $R_{i\rightrightarrows}$. If B_i does not default, then it repays all of its liabilities in full: $R_{i\to j} = F_{i\to j}$ for all j. But B_i can default. In that case, its repayments are less than the face values of its liabilities. We assume that banks repay their interbank

[5]Throughout, we use "liquidity" in the sense of "inside liquidity" in Holmström and Tirole (1998).

[6]The formulation can be interpreted literally in terms of private benefits or cash diversion; for microfoundations in terms of other agency problems, see, e.g., DeMarzo and Fishman (2007) and Donaldson, Gromb, and Piacentino (2021a).

[7]For example, in 2021, Barclays PLC's net interbank position was about an eighth of its gross, Lloyd's about a fourteenth, and HSBC's about a fifth. Specifically, their loans to and from other banks were, respectively, about 13.9 and 16.4, 7.0 and 7.6, and 83.1 and 101.1 billion GBP; see home.barclays/content/dam/home-barclays/documents/investor-relations/reports-and-events/annual-reports/2021/Barclays-Bank-PLC-2021-AR.pdf, p. 207, lloyds-bankinggroup.com/assets/pdfs/investors/annual-report/2021/2021-lbg-annual-report.pdf, pp. 207–208, and hsbc.com/investors/results-and-announcements/annual-report, p. 310.

liabilities in pro rata shares $F_{i\to j}/F_{i\rightrightarrows} =: \hat{F}_{i\to j}$:

$$R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}. \tag{1}$$

This assumption follows the literature (Eisenberg and Noe (2001)) and reflects bankruptcy law and practice.[8]

At Date 1, after the shocks $\boldsymbol{\sigma}$ are realized, banks can borrow in a competitive market. We assume that they can issue new liabilities of high priority; existing interbank liabilities are thereby diluted, in that their claims on assets are now subordinated to new creditors'.[9] These new liabilities could represent repos, which have "super-senior" claims on assets in bankruptcy.[10,11] As a result, they can borrow against all their pledgeable assets—the pledgeable part θy of their long-term assets and their interbank claims $R_{i\leftleftarrows}$. B_i is thus liquidated if the total value of its pledgeable assets is less than its liquidity needs, or if

$$\theta y + R_{i\leftleftarrows} < \ell\sigma_i. \tag{2}$$

This captures our key twist relative to the literature: The long-term liabilities $F_{i\rightrightarrows}$ that B_i has in place do not appear. As they can be diluted, they do not impede B_i from raising liquidity, whereas, in contrast, short-term liabilities do (see Section 3).

In liquidation, a bank's payoff and repayments are zero.[12]

[8]Csóka and Herings (2021) provides an axiomatic foundation for the pro rata assumption.

[9]Note that dilution, as used here, need not decrease the value of existing liabilities. Although existing liabilities have a smaller slice of the asset pie, these assets might be more valuable if dilution increases the size of the pie (e.g., meeting a liquidity shock).

[10]As such, the model could reflect how banks suffering liquidity shocks use super-senior (repo) financing to relax their borrowing constraints, something LTCM, Bear Stearns, and Lehman Brothers all did (or tried to do). (See, e.g., Jorion (2000, pp. 282–284) on LTCM, Rose, Bergstresser, and Lane (2009, pp. 11–13) on Bear, and Valukas (2010 , pp. 3 and 9–10) on Lehman.)

[11]Thus, there are two priority classes of debt in our model: new repo-type debt paid first and interbank debts paid next pro rata. This is a good approximation of reality, in which there are two main priority classes: secured debt paid first and unsecured paid next pro rata (see, e.g., Schwartz (1989)). (AOT also features two priority classes, but the senior debt is in place at inception.)

[12]The assumption that interbank repayments are zero follows AOT, in which liquidity shocks—their "outside obligations"—are senior to interbank debts. The opposite assumption makes the repayment in equation (5) discontinuous and can lead to non-existence of the payment equilibrium (see Definition 1 below).

The non-pledgeable assets $(1-\theta)y$ are destroyed. That is the only deadweight loss in the model (except in extensions; see Section 6.1 and Section 6.2).

Banks that are not liquidated at Date 1 continue to produce y at Date 2. B_i's total (real and financial) assets are thus $y - \ell\sigma_i + R_{i\leftleftarrows}$. (The cash raised via new debt at Date 1 and its associated repayment do not appear because, the liability being riskless, the amount borrowed equals the amount repaid, and they cancel out at Date 2.) At this point, B_i can either default and capture its non-pledgeable asset value $(1-\theta)y$ or repay in full. It thus defaults if

$$\theta y - \ell\sigma_i + R_{i\leftleftarrows} < F_{i\rightrightarrows}. \tag{3}$$

Observe that, unlike liquidation, which destroys non-pledgeable assets, default alone does not cause a deadweight loss, but just a transfer from creditors to debtors. (We include deadweight losses from default in Section 6.2.)

Combining the liquidation condition (2) and the default condition (3), we have the sequentially rational repayment:

$$R_{i\rightrightarrows} = \begin{cases} 0 & \text{if } \theta y - \ell\sigma_i + R_{i\leftleftarrows} \le 0\,, \\ \theta y - \ell\sigma_i + R_{i\leftleftarrows} & \text{if } \theta y - \ell\sigma_i + R_{i\leftleftarrows} \in (0, F_{i\rightrightarrows}]\,, \\ F_{i\rightrightarrows} & \text{otherwise} \end{cases} \tag{4}$$

$$= \max\Big\{ 0\,,\, \min\big\{ \theta y - \ell\sigma_i + R_{i\leftleftarrows}\,,\, F_{i\rightrightarrows} \big\} \Big\}. \tag{5}$$

To define the equilibrium, we also require that markets clear: The repayments B_i receives

It does not, however, alter our central insights on the benefits of high indebtedness and connectedness with long-term debt; see Donaldson and Piacentino (2018).

from other banks coincide with the repayments other banks make to it:

$$R_{i\leftleftarrows} = \sum_{j \neq i} R_{j \to i}. \tag{6}$$

Definition 1 (Payment equilibrium)**.** *A* payment equilibrium *is a repayment vector* $\{R_{i \to j}\}_{i \neq j}$ *for each state* $\boldsymbol{\sigma}$ *such that the repayments*

(i) are sequentially rational (equation (5)),

(ii) are paid pro rata (equation (1)), and

(iii) clear the market (equation (6)).

It is convenient to combine the equilibrium conditions to write a vector fixed point equation:

$$\mathbf{R}_{\rightrightarrows} = \left[\min \left\{ \mathbf{F}_{\rightrightarrows}\, , \, \theta y \mathbf{1} - \ell \boldsymbol{\sigma} + \hat{\mathbf{F}}^{\top} \mathbf{R}_{\rightrightarrows} \right\} \right]^{+}, \tag{7}$$

a solution of which is often called the "clearing vector." ($\mathbf{1}$ denotes the vector of N ones: $(1, ..., 1) \in \mathbb{R}^N$.)

The only deadweight loss in the model is due to liquidation. Thus we adopt the following notion of efficiency:

Definition 2 (Efficiency)**.** *One network is more* efficient *than another if fewer banks are liquidated in equilibrium for every state* $\boldsymbol{\sigma}$*.*

This is a strengthening of AOT's notions of "stability" (fewer liquidations on average for a given number of shocks) and "resilience" (fewer liquidations in the worst case scenario) in that if one network is more efficient than another it is more stable and more resilient too. The efficiency ranking is not a total order on the set of networks, but we derive strong enough results that it suffices for our purposes (except in the extension with heterogeneous banks in Section 6.3, where we modify it; see Definition 11).

For several of our results, it is useful to define the following network structures:

Definition 3 (Network typology)**.** *A network is* regular *if* $F_{i\rightrightarrows} = F_{j\rightrightarrows}$ *for* $i \neq j$*,* symmetric *if* $F_{i\to j} = F_{j\to i}$ *for* $i \neq j$*,* complete *if* $F_{i\to j}$ *is constant for* $i \neq j$*,* ring *if it is regular and* $F_{i\to j} = 0$ *unless* $j = i + 1$ *(mod* N*).*

In words, in a regular network, each bank has the same total liabilities; in a symmetric network, each pair of banks has zero net positions; in a complete network, every bank has the same liability to every other; in a ring network, each bank has liabilities to one other and claims one other in a circle.

It is also useful to define several properties of networks, capturing how closely banks are connected to one another.

Definition 4 (Delta connectedness)**.** *A regular network* $\mathbf{F}$ *is* δ*-connected if there is a subset of banks* $\mathscr{B}$ *such that* $\hat{F}_{i\to j} \leq \delta$ *and* $\hat{F}_{j\to i} \leq \delta$ *for all* $i \in \mathscr{B}$ *and* $j \in \mathscr{B}^c$*.*

It is connected *if it is not* δ*-connected for* $\delta = 0$*.*

In words, a network has low δ if one of its components has weak ties to the rest of it.

Definition 5 (Harmonic distance)**.** *For a regular network* $\mathbf{F}$*, the* harmonic distance *from* B_i *to* B_j *is the solution to* $d_{i\to j} := 1 + \sum_{k\neq i} d_{i\to k}\hat{F}_{k\to j}$ *for* $i \neq j$ *and* $d_{i\to i} = 0$*.*

In words, $d_{i\to j}$ is the liability-weighted distance from $i \to j$, which captures how easily liquidity (or the lack thereof) can flow from B_i to B_j.

Definition 6 (Bottleneck parameter)**.** *For a regular network* $\mathbf{F}$*, the* bottleneck parameter *is*

$$\beta = \min_{\mathscr{B}} \sum_{i\in\mathscr{B}} \sum_{j\in\mathscr{B}^c} \frac{\hat{F}_{i\to j}}{|\mathscr{B}||\mathscr{B}^c|}. \tag{8}$$

In words, a network has low β if one of its components has relatively low liabilities to the rest of it. It is similar to δ above, but directional (in that it is silent about the liabilities from the rest to the component).

3 Short-term Debt Benchmark

Here we consider the network in which the interbank liabilities $F_{i\to j}$ are due at Date 1 instead of Date 2. This benchmark helps us both to compare our model to the literature and to contrast our results to those therein.

Now interbank liabilities, being due immediately, cannot be diluted with new debt at Date 1. Thus B_i is liquidated if its pledgable assets are insufficient to cover not only its liquidity needs $\ell\sigma_i$ but also its interbank liabilities $F_{i\rightrightarrows}$, or

$$\theta y + R_{i\leftleftarrows} < \ell\sigma_i + F_{i\rightrightarrows}. \tag{9}$$

This condition for liquidation coincides with that for default. In that respect, the benchmark contrasts with the baseline, in which the conditions are different. Nonetheless, the equations for the clearing vector are the same in both versions, as the liquidation value at Date 1 coincides with the pledgeable value at Date 2: Banks that would default at Date 2 in the baseline model make the same repayment when they are liquidated at Date 1 here. Thus we do not need to adjust the equilibrium definition here. Only efficiency changes.

Without the distinction between liquidation and default, this benchmark boils down to AOT's model:

Lemma 1 (Isomorphism between benchmark and AOT)**.** *There is an isomorphism between equilibrium and efficiency in our short-term debt benchmark and AOT's model in the case in which long-term assets are fully destroyed by default and cash holdings are zero (under which they derive most of their results).*

(The isomorphism between our benchmark and AOT, while mechanical, was unexpected to us, as the models seemed different prima facie. Our model seemed to be about liquidity, theirs about solvency. We now see both as about both: In both, a shock decreases total asset value (solvency) and might not be met only because long-term assets are not fully pledgeable (liquidity).)

We now restate, and sometimes strengthen, several of AOT's results, focusing on those that contrast with our results on the long-term debt network, starting with debt levels.

Lemma 2 (Netting in short-term benchmark)**.** *Suppose* $\mathbf{F}$ *is a regular network.* $\alpha\mathbf{F}$ *is less efficient than* $\mathbf{F}$ *whenever* $\alpha > 1$.

This is a generalization of AOT's Proposition 3 (p. 574) to an arbitrary number of shocked banks (they prove it for just one). It says that less debt is a good thing. Indeed, it would be better to have none whatsoever ($\alpha = 0$). Intuitively, when one bank defaults on its liability to another, the other finds it harder to pay its liability to yet another. So distress propagates from shocked banks to otherwise healthy ones, especially when debts are high ($\alpha > 1$).

We now turn to network connectedness.

Lemma 3 (Delta connectedness in short-term benchmark)**.** *Suppose* ℓ *is sufficiently large and exactly one bank is shocked* ($|\boldsymbol{\sigma}| = 1$)*.*

(i) The ring network is the least efficient among all regular networks with $F_{i\rightrightarrows} > (N-1)\theta y$*.*

(ii) Any regular δ*-connected network with* $N\delta < \theta y / F_{i\rightrightarrows}$ *is strictly more efficient than the ring.*

This strengthens some of the statements in AOT's Proposition 6 (p. 577–578) by adapting them to our notion of efficiency (Definition 2). It says that less connectedness is a good thing. The ring network, in which every bank has a large exposure to another, is the worst in a class. It is better to weaken the exposures in the sense of lowering delta connectedness. Intuitively, δ captures how much risk can transmit between two components, so, unlike in the ring network, risk cannot spillover from one to another when δ is low.

We now show that there is a "default radius" around a shocked bank.

Lemma 4 (Default radius in short-term benchmark)**.** *Let* $\mathbf{F}$ *be a regular network with* $F_{i\rightrightarrows} \equiv F$ *and suppose that exactly one bank, say* B_j*, is shocked and does not meet its liquidity shocks* ($R_{j\rightrightarrows} = 0$)*. Define* $d^{ST} := \frac{F}{\theta y}$*.*

(i) If $d_{j\to i} < d^{ST}$, then B_i is liquidated.

(ii) If all banks are liquidated, then $d_{j\to i} < d^{ST}$ for all i.

This is AOT's Proposition 8 (p. 579). It says that the harmonic distance d is, in a sense, the right measure of one bank's exposure to another, in that it captures exactly whether a shocked bank's distress will transmit to an otherwise healthy bank through the network. It defines a radius around a shocked bank within which all banks are liquidated.

AOT link the harmonic distance d to the bottleneck parameter β using Markov chains. They show, roughly, that $d_{i\to j}$ is the mean hitting time of a Markov chain from state i to j and that the bottleneck parameter is closely related to the "conductance" of a graph, which measures how hard it is for a Markov chain on a graph to leave a set of nodes. Hence the next result:

Lemma 5 (Bottleneck connectedness in short-term benchmark)**.** *Suppose the conditions of Lemma 4 are met and that, additionally, the network* $\mathbf{F}$ *is symmetric. Define* $\beta^{ST} := 4\sqrt{\frac{\theta y}{NF}}$ *and* $\beta_{ST} := \min\left\{\frac{\theta y}{2NF}, 1\right\}$.

(i) If $\beta > \beta^{ST}$, then all banks are liquidated.

(ii) If $\beta < \beta_{ST}$, then at least one bank is not liquidated.

This is AOT's Corollary 2 (p. 581). It captures the idea that if all banks are closely connected then risk is so easily transmitted to other banks that a shock at one can lead all to fail, whereas if they are not, it cannot. Specifically, if at least two components are not closely connected, so β is small, risk in one of them cannot spread to the other.

4 Properties of Long-term Debt Networks

We now turn to the properties of networks in our baseline model, with long-term debt. For each result in our short-term debt benchmark, we prove a counterpart with opposite sign:

Whereas indebtedness and connectness do harm with short-term debt, they do good with long-. The reason is that rather than enabling the liquidity shortage to spread from shocked banks, they allow the liquidity surplus to spread from healthy ones.

We start with existence and uniqueness.

Proposition 1 (Existence and uniqueness)**.** *For any network* $\mathbf{F}$*, a payment equilibrium exists and is generically unique.*

We now turn to debt levels.

Proposition 2 (Netting)**.** *Suppose* $\mathbf{F}$ *is a regular network.* $\alpha\mathbf{F}$ *is more efficient than* $\mathbf{F}$ *whenever* $\alpha > 1$.

This is the counterpart of Lemma 2. It says that *more* debt is a good thing.[13] The reason is that high debts ($\alpha > 1$) allow shocked banks to borrow more from the market and weather shocks. To see why, suppose two banks, B_i and B_j, have perfectly off-setting debts, owing each other the same amount: $\alpha F_{i\to j} = \alpha F_{j\to i} = \alpha F$. If one of them, say B_i, suffers a liquidity shock, it raises liquidity in the market by pledging its own assets and the claim it has from B_j, raising $\theta y + R_{j\to i}$. The larger B_i's claim $\alpha F_{j\to i}$ on B_j is, the higher its value $R_{j\to i}$ is, and the more liquidity it can raise.

Yes, increasing α gives B_i a larger liability to repay in addition to a larger claim to pledge. But it need not repay it. It has the option to default on it. That creates another option, to dilute it. It can pledge assets to new liabilities (dilution) leaving the existing liability empty handed (default).

That allows it to raise liquidity to the extent that the assets backing the new liabilities are valuable. And B_i's assets are valuable if its claim on B_j is, or $R_{j\to i}$ is high, which it is whenever B_j is not shocked. In that case, B_j generally repays in full: $R_{j\to i} = \alpha F$. (Not

[13] Although, per the result, increasing debt cannot hurt if the relative debts stays the same—if $\alpha\mathbf{F}$ increases but $\mathbf{F}$ stays constant—it can if they change—i.e. if some debts increase and some do not—as illustrated by how making a network "more symmetric" can decrease efficiency (in Section 5).

being shocked, B_j prefers not to exercise its default option, which renders the dilution option moot (cf. equation (3)).[14]

B_j suffers when B_i is shocked and it is not, i.e. when it makes its repayment and B_i exercises its default-dilution option. But it benefits in the symmetric case, i.e. when it exercises its option and B_i does not. Liquidity gets transferred from the bank with liquidity to the bank that needs it: Zero-net long-term debt has positive net present value. Per the result, increasing debt increases overall efficiency.[15]

That is not so in the short-term debt benchmark (and most other models), in which zero-net debts have zero NPV at most (Lemma 2). The reason is that short-term debt, being due right away, cannot be diluted: The claims on the left side of bank balance sheets increase debt capacity but the liabilities on the right decrease it; they cancel each other out at best.

We turn to network connectedness next.

Proposition 3 (Delta connectedness)**.** *Suppose ℓ is sufficiently large and exactly one bank is* not *shocked* ($|\mathbf{1} - \boldsymbol{\sigma}| = 1$).

(i) The ring network is the most efficient among all regular networks with $F_{i\rightrightarrows} > \theta y$.

(ii) For any δ, there is a δ-connected network that is strictly less efficient than a ring.

This is the counterpart of Lemma 3. It says that *more* connectedness is a good thing. The ring network, in which every bank has a large exposure to another, is the best in a class (given a single shocked bank). It is better not to weaken interbank exposures in the sense that making δ small could lead to a strictly worse outcome. Intuitively, δ captures how much liquidity can transmit between two components, so when δ is low, liquidity cannot flow from banks that have it to banks that need it.

[14] This mechanism is self-enforcing, in that, as banks cannot avoid being diluted, they are committed ex ante to transfers they would prefer not to make ex post. Leitner (2005) uncovers another self-enforcing mechanism to transfer liquidity in financial networks: Healthy banks commit to transfer liquidity to distressed ones by exposing themselves to their default through the network.

[15] Efficient "dilutable debt" also appears in Diamond (1993), Donaldson, Gromb, and Piacentino (2020, 2021b), and Hart and Moore (1995).

We turn to the long-term-debt counterpart to short-term debt's default radius around a shocked bank. It is a "salvation radius" around a healthy bank.

Proposition 4 (Salvation radius)**.** *Let* $\mathbf{F}$ *be a regular network with* $F_{i\rightrightarrows} \equiv F$ *and suppose that exactly one bank, say* B_j*, is* not *shocked and that it does not default. Define* $d^{LT} := \frac{F}{\ell - \theta y}$*.*

(i) If $d_{j\to i} < d^{LT}$*, then* B_i *is not liquidated.*

(ii) If no bank is liquidated, then $d_{j\to i} < d^{LT}$ *for all* i*.*

This is the counterpart of Lemma 4. It says that the harmonic distance captures not only how defaults transmit from shocked to healthy banks in short-term debt networks, but also how the option to dilute allows liquidity to flow from healthy to shocked banks in long-term ones. It defines a radius around a not-shocked bank within which *no* bank is liquidated.

AOT's link between the harmonic distance and the bottleneck parameter also applies to our long-term debt network—it relies on only network structures, not equilibrium behavior. Hence we have the next result:

Proposition 5 (Bottleneck connectedness)**.** *Suppose the conditions of Proposition 4 are met and that, additionally, the network* $\mathbf{F}$ *is symmetric. Define* $\beta^{LT} := 4\sqrt{\frac{\ell-\theta y}{NF}}$ *and* $\beta_{LT} := \min\left\{\frac{\ell-\theta y}{2NF}, 1\right\}$*.*

(i) If $\beta > \beta^{LT}$*, then no bank is liquidated.*

(ii) If $\beta < \beta_{LT}$*, then at least one bank is liquidated.*

This is the counterpart of Lemma 5. It captures the idea that if all banks are closely connected, one bank's excess liquidity can flow through the system to save all banks and, conversely, if they are not, it cannot. Specifically, if at least two components are not closely connected (β is small), the banks in one can raise little liquidity by diluting their liabilities to banks in the other. What they can raise can be so limited that they end up unable to save themselves from liquidation.

5 Efficiency and the Exponential Network

Here we define constrained efficiency and construct a class of networks—the "exponential networks"—that implement it. We conclude the section with an example contrasting complete and exponential networks, which captures the main ideas.

Definition 7 (Planner's problem and constrained efficiency)**.** *The* planner's problem *is to find a set of transfers* $\{t_i\}_i$ *for each* $\boldsymbol{\sigma}$*, with* t_i *paid to each* B_i*, to minimize the number of liquidated banks* $\left|\left\{i : \theta y + t_i - \ell\sigma_i < 0\right\}\right|$ *subject to each bank's liquidity constraint* $t_i \geq \min\{\ell\sigma_i - \theta y, 0\}$ *and to liquidity being conserved* $\sum t_i \leq 0$*.*

A network is constrained efficient *if the equilibrium is no less efficient than the planner's solution.*

In words, the planner wants to minimize the number of liquidated banks by transferring liquidity within the system. It must respect the limited pledgeability friction—it cannot raise more from any one bank than the net liquidity it has (that is no more than θy from a not-shocked bank and zero from a shocked one, per the constraint $t_i \geq \min\{\ell\sigma_i - \theta y, 0\}$, given $\ell > \theta y$). The next result characterizes the solution.

Lemma 6 (Constrained efficiency)**.** *A network is constrained efficient if the number of liquidated banks is*

$$L^* := \max\left\{ 0\,,\ \left\lceil \frac{S\ell - N\theta y}{\ell - \theta y} \right\rceil \right\} \tag{10}$$

for each state $\boldsymbol{\sigma}$*, where* S *denotes the number of shocked banks.*

It turns out that the social planner should generally raise as much liquidity as possible from each not-shocked bank, levying the tax $-t_i = \theta y$ if $\sigma_i = 0$, and transfer shocked banks either just enough liquidity to survive or none at all, i.e., either $t_i = \ell - \theta y$ or $t_i = 0$ if $\sigma_i = 1$. Using S and L to denote the numbers of shocked and liquidated banks, the planner's budget constraint says that the total subsidy—the transfer $\ell - \theta y$ to each of the $S - L$ shocked banks that is not liquidated—must be less than the total tax—the transfer θy from each of the $N - S$ banks that is not shocked:

$$(S - L)(\ell - \theta y) \leq (N - S)\theta y. \tag{11}$$

Solving for the smallest positive integer L that satisfies the above gives the result.

That argument points to two key properties of the planner's solution, both of which help avoid "wasting liquidity":

(i) It extracts the maximum tax from not-shocked banks, since they are not liquidated anyway.

(ii) It gives nothing to liquidated banks, since, analogously, they are liquidated anyway.

We aim to construct a network with both properties (Proposition 6 below). We now build up to it in steps, showing how to achieve one and then the other, starting with the first:

Lemma 7 (High debt mutualizes assets)**.** *Let* $\mathbf{F}$ *be a connected network. If* α *is sufficiently large, then in the equilibrium of* $\alpha\mathbf{F}$ *either (i) all not-shocked banks make the maximum net payment,* $R_{i\rightrightarrows} - R_{i\leftleftarrows} = \theta y$*, or (ii) no bank is liquidated.*

This says that if debts are sufficiently high and liquidity is sufficiently scarce (in the sense that at least one bank is liquidated), then each not-shocked bank provides the maximum amount of liquidity. Intuitively, to increase interbank debts is to make each bank's assets a larger fraction of others' balance sheets—it "mutualizes" the banking system, making each bank more like the whole system.[16,] As a result, when liquidity is scarce overall, no surviving bank retains excess liquidity. In contrast, if liquidity is not scarce then no bank is liquidated if debt levels are sufficiently high:

[16]This role of default in facilitating a socially efficient transfer of liquidity contrasts with the literature, in which it typically constitutes a social cost, facilitating rent extraction at best (see Farboodi (2021) and Perotti and Spier (1993)).

Corollary 1 (First best with high debt for small shocks)**.** *Let* $\mathbf{F}$ *be a connected network and suppose that* $N\theta y > S\ell$ *(i.e.* $L^* = 0$*). If* α *is sufficiently large, then no bank is liquidated in the equilibrium of* $\alpha\mathbf{F}$*.*

The second property—that liquidated banks are transferred nothing—points to how the planner's allocation is necessarily discriminatory: It prioritizes some shocked banks over others. (If in contrast, it allocated the excess liquidity equally among all shocked banks it ends up saving none of them unless it can save them all. As we illustrate in an example below, that symmetry makes the complete network "robust yet fragile," per AOT's result.) That suggests that whenever liquidity is scarce (S is large) a network must be asymmetric to be (constrained) efficient. The following definitions characterize ways in which a network can be asymmetric.

Definition 8 (Assortativity)**.** *A network* $\mathbf{F}$ *is* assortative *if*

$$F_{i\to k} > F_{i\to l} \implies F_{j\to k} > F_{j\to l} \tag{12}$$

for all distinct $i, j, k,$ *and* l*.*

In words, a network is assortative if whenever B_i owes more to one bank than another, so does B_j. Assortativity allows us to rank banks unambiguously by the size of their interbank liabilities. The next definition quantifies/controls that ranking.

Definition 9 (s-dominance)**.** *For a network* $\mathbf{F}$*,* B_i*'s liabilities are* s-dominated *for* $s \in (0, 1)$*, if there is a permutation* π_i *on* $\{1, ..., N\}$ *with* $\pi_i(i) = i$*, such that*

$$\frac{F_{i\to\pi_i(j+k)}}{F_{i\to\pi_i(j)}} \leq s^k \tag{13}$$

for all j *and* $k \geq 0$ *such that* $j \neq i$ *and* $j + k \neq i$*.*

In words, for $s < 1$, B_i's liabilities to others decay rapidly—its second largest liability is only at most a fraction s of its largest, and so on. (The permutation π_i in the definition just ranks the debts by size.)

Together the definitions above define what we call exponential networks:

Definition 10 (Exponential networks)**.** *A network is an* exponential network *(with base s) if it is connected, it is assortative, and B_i's debts are s-dominated for all i.*

In words, every bank's debt to B_2 is only at most a fraction s of its debt to B_1 and so on—the permutation π_i in Definition 9 ranks each bank's creditors the same way (assumed w.l.o.g. to be the same as their index ordering per Definition 8). The exponential network generates an approximately exponential distribution of bank asset size, $y + F_{i\rightrightarrows}$.[17,18]

The next two results characterize the payments made to liquidated banks in an exponential network.

Lemma 8 (Controlling relative payments to liquidated banks)**.** *Suppose* $\mathbf{F}$ *is an exponential network with base s and let B_{i^*} be the largest liquidated bank. For each other liquidated bank B_j,*

$$R_{j\leftleftarrows} \leq s^{j-i^*} R_{i^*\leftleftarrows}. \tag{14}$$

This says that when debt levels in the network are exponentially controlled (given s-dominance), so are the repayments to liquidated banks in equilibrium.

The previous result says that s controls the relative payments among liquidated banks. The next says that it controls the total payment to all of them.

Lemma 9 (Controlling total payments to liquidated banks)**.** *Suppose* $\mathbf{F}$ *is an exponential network with base s. If at least one bank is liquidated in equilibrium, then*

$$\sum_{i\in\mathscr{L}} R_{i\leftleftarrows} < \frac{\ell - \theta y}{1 - s}, \tag{15}$$

[17] The network thus captures the empirical fact that bank size decays rapidly. That is often modeled with a Pareto distribution, which fits the distribution of large banks well. Small banks are smaller than it predicts, however, in line with the exponential distribution (Janicki and Prescott (2006)).

[18] Another paper in which intermediation networks give rise to an endogenous bank size distribution is Farboodi, Jarosch, and Shimer (2017).

where $\mathscr{L}$ denotes the set of liquidated banks.

This says that the total transfer to liquidated banks can be made arbitrarily close to a single shocked bank's liquidity shortfall, $\ell - \theta y$, by making s sufficiently small. Intuitively, for small s, the liquidity wasted by transferring it to banks that end up being liquidated anyway would barely have been enough to save even a single one of them (per property (ii) above).

To sum up, if debts are large then no liquidity is wasted on banks that would not have been liquidated anyway (Lemma 7) and if the network is exponential then little is wasted on those that would have been (Lemma 9). The next result builds on these findings to show how to achieve constrained efficiency.

Proposition 6 (Efficiency of exponential networks)**.** *Define*

$$s^* := 1 - \frac{\ell - \theta y}{N\theta y - S\ell + (1 + L^*)(\ell - \theta y)}. \tag{16}$$

Let $\mathbf{F}$ *be an exponential network with base* $s \leq s^*$*. For* α *sufficiently large,* $\alpha\mathbf{F}$ *is constrained efficient as long as* $\frac{S\ell - N\theta y}{\ell - \theta y}$ *is not an integer.*

In words, an exponential network with a rapidly decaying distribution of debts is optimal in all but a knife-edge case (i.e. all but the case in which $\frac{S\ell - N\theta y}{\ell - \theta y}$ is an integer).

This result has a limitation when achieving constrained efficiency requires almost all of the liquidity available, i.e. when the slack in inequality (11) becomes small for $L = L^*$ ($(N - S)\theta y - (S - L^*)(\ell - \theta y) \to 0$). In this case, s^* becomes small so the largest bank in the exponential network is many times larger than the second largest one, and so on. Moreover, in the limit when it equals zero, the network does not achieve constrained efficiency—that is the integer case from the proposition. But the next result suggests this limitation is not too worrisome if we accept a weaker notion of efficiency:

Proposition 7 (Approximate efficiency of exponential networks)**.** *Let* $\mathbf{F}$ *be an exponential network with base* $s \leq 1/2$*. For* α *sufficiently large,* $\alpha\mathbf{F}$ *is "almost constrained efficient" in that at most* $L^* + 1$ *banks are liquidated.*

Finally, we point out that even in the knife-edge case in which the exponential network does not achieve constrained efficiency, no other connected network does either.

Lemma 10 (Inefficiency of other networks)**.** *Suppose $\frac{S\ell - N\theta y}{\ell - \theta y}$ is an integer and $S\ell > N\theta y$ and that $\mathbf{F}$ is fully connected in that $F_{i\to j} > 0$ for all $i \neq j$. The equilibrium is not constrained efficient.*

This result suggests that the exponential network is the "best" no matter the parameters.[19]

Exponential network example. Finally, we consider an example to illustrate (i) how a complete network leads to inefficient liquidation by allocating liquidity to banks that end up being liquidated in equilibrium and (ii) how an exponential network solves the problem.

The illustration requires that shocks are large enough that at least one bank is liquidated in the constrained-efficient outcome (otherwise a complete network can save all banks). Hence we consider three banks, two of which are shocked: There are $N = 3$ banks with assets $y = 2$, a fraction $\theta = 1/2$ of which is pledgeable. Exactly two banks suffer shocks, $S \equiv \sum \sigma_i = 2$, of size ℓ. We assume that $\ell = 8/5$ so that in the constrained-efficient outcome exactly one bank is liquidated in each state: $\ell < 3\theta y < 2\ell$ (i.e. $8/5 < 3 < 16/5$). Observe that covering either shocked bank's liquidity shortfall requires at least 3/5 of the other's surplus: $\ell - \theta y = 3\theta y/5$.

We start with the complete network as benchmark and show that both shocked banks are always liquidated. Then we illustrate how the exponential network saves one of them, achieving the constrained-efficient outcome.

Complete network benchmark. Suppose each bank has total liablities $F_{i\rightrightarrows} \equiv F$ to others ($F/2$ to each of the other two). For each state $\boldsymbol{\sigma}$, equation (5) gives the system the clearing vector $\mathbf{R}_{\rightrightarrows} = [R_{i\rightrightarrows}]_i$ must solve (the full matrix of equilibrium repayments is then given by the pro rata shares $R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}$ by equation (1)). When the shocked banks are B_1 and

[19] As the proof of Lemma 10 implies, a network that is not fully connected could do better for some realizations of $\boldsymbol{\sigma}$ but not for all.

B_2, the system is:

$$\begin{cases} R_{1\rightrightarrows} = \max\left\{ 0\,,\, \min\left\{ -\tfrac{3}{5} + \tfrac{1}{2}R_{2\rightrightarrows} + \tfrac{1}{2}R_{3\rightrightarrows}\,,\, F \right\} \right\}, \\ R_{2\rightrightarrows} = \max\left\{ 0\,,\, \min\left\{ -\tfrac{3}{5} + \tfrac{1}{2}R_{1\rightrightarrows} + \tfrac{1}{2}R_{3\rightrightarrows}\,,\, F \right\} \right\}, \\ R_{3\rightrightarrows} = \max\left\{ 0\,,\, \min\left\{ \ 1 + \tfrac{1}{2}R_{1\rightrightarrows} + \tfrac{1}{2}R_{2\rightrightarrows}\,,\, F \right\} \right\}. \end{cases} \tag{17}$$

The first two lines of the system illustrate the problem with the complete network. The repayments from the bank with excess liquidity, B_3, are allocated equally between the two banks with a liquidity shortfall, B_1 and B_2—each gets $\frac{1}{2}R_{3\rightrightarrows}$ (plus a symmetric transfer from the other shocked bank, $\frac{1}{2}R_{2\rightrightarrows}$ or $\frac{1}{2}R_{1\rightrightarrows}$). But each bank needs more than that to survive. Allocating scarce resources equally means that no one has enough.

As the system is symmetric, the problem is analogous when the other pairs of banks are shocked. Solving it gives $R_{i\rightrightarrows} = 0$ if $\sigma_i = 1$ and $R_{i\rightrightarrows} = \min\{F, 1\}$ otherwise, implying that both shocked banks are always liquidated (equation (4)).

Exponential network. Now we turn to an exponential network with base $s = 1/2$:

$$\mathbf{F} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}. \tag{18}$$

Note that the off-diagonal entries in each row and column are decreasing (assortativity per Definition 8) and that each is at most a fraction 1/2 of the previous (s-dominance per Definition 9). We can compute each bank's total liabilities $F_{i\rightrightarrows}$ (the row sums of $\mathbf{F}$) and the fraction of its payments it makes to each other bank ($\mathbf{F}$ normalized by $F_{i\rightrightarrows}$) :

$$\mathbf{F}_{\rightrightarrows} = \begin{bmatrix} 3 \\ \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \quad \& \quad \hat{\mathbf{F}} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{4}{5} & 0 & \frac{1}{5} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}. \tag{19}$$

It turns out that no matter what pair of banks is shocked, only one is liquidated. To see why, consider the case in which B_1 and B_2 are shocked. In that case, the clearing vector solves (substituting from equation (19) into equation (5)):

$$\begin{cases} R_{1\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ -\frac{3}{5} + \frac{4}{5}R_{2\rightrightarrows} + \frac{2}{3}R_{3\rightrightarrows}\,,\, 3 \big\} \Big\}, \\ R_{2\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ -\frac{3}{5} + \frac{2}{3}R_{1\rightrightarrows} + \frac{1}{3}R_{3\rightrightarrows}\,,\, \frac{5}{2} \big\} \Big\}, \\ R_{3\rightrightarrows} = \max\Big\{ 0\,,\, \min\big\{ \ 1 + \frac{1}{3}R_{1\rightrightarrows} + \frac{1}{5}R_{2\rightrightarrows}\,,\, \frac{3}{2} \big\} \Big\}. \end{cases} \tag{20}$$

The first two lines of the system illustrate how the exponential network allocates liquidity efficiently. The repayments from B_3, the bank with excess liquidity, are allocated primarily to one of the two banks with the liquidity shortfall—B_1 gets $\frac{2}{3}$ of B_3's total repayment, B_2 only $\frac{1}{3}$ of it. And that allows B_1 to survive. B_2 is liquidated. But that is (constrained) efficient as there are not enough resources to save them both anyway.

Solving the system gives the clearing vector $\mathbf{R}_{\rightrightarrows} = \left(\frac{3}{35}, 0, \frac{36}{35}\right)$, which, having only one zero entry, affirms that only B_3 is liquidated (equation (4)).

The other cases are analogous. When B_1 and B_3 are shocked the clearing vector is $\mathbf{R}_{\rightrightarrows} = \left(\frac{3}{7}, \frac{9}{7}, 0\right)$, implying only B_3 is liquidated, and when B_2 and B_3 are shocked it is $\mathbf{R}_{\rightrightarrows} = \left(\frac{39}{35}, \frac{1}{7}, 0\right)$, implying only B_3 is.

6 Extensions

Under our baseline assumptions (i) liquidation is always inefficient and (ii) default absent liquidation is costless. Here we relax these assumptions (albeit only for specific networks). We show how to choose debt levels to balance the benefit of high debt in providing liquidity to avoid inefficient liquidation per the baseline with the cost of inducing excessive continuation/default included here.

We also relax the assumption that all banks are identical. Using results from computer science, we show conditions under which the exponential network implements/approximates the constrained efficient outcome in this case.

6.1 Risky Assets and Too Few Liquidations

So far, we assumed that y was sufficiently large that liquidation was always inefficient. Now we assume that y can have any value (but is the same for all banks). Thus it is efficient for all shocked banks to be liquidated if y is low but not if it is high. Here we denote the threshold below which liquidation is efficient by y^* and we show how to choose debt levels in a complete network to implement the efficient liquidation policy.[20]

We consider a complete network with debt levels F, focusing on the case in which $S\ell < N\theta y$ for all y (so, in principle, no bank need be liquidated: $L^* = 0$ in Lemma 6). As the network is symmetric, each shocked/not-shocked bank makes and receives the same payments; we index all shocked banks' payments by s and not-shocked banks' by n. From equation (7), the payment to each type is a pro rata share of the repayment made by all other banks of that type and by all banks of the other type. Hence the equilibrium equations for any y are:

$$\begin{cases} R_{s\rightrightarrows} = \left[\min\left\{ F\,,\, \theta y - \ell + \dfrac{1}{N-1}\Big((S-1)R_{s\rightrightarrows} + (N-S)R_{n\rightrightarrows}\Big)\right\}\right]^+, \\[2ex] R_{n\rightrightarrows} = \left[\min\left\{ F\,,\, \theta y + \dfrac{1}{N-1}\Big(SR_{s\rightrightarrows} + (N-S-1)R_{n\rightrightarrows}\Big)\right\}\right]^+. \end{cases} \tag{21}$$

Solving gives the equilibrium repayments:

$$\begin{cases} R_{s\rightrightarrows} &= \left[F - \frac{N-1}{N-S}(\ell - \theta y)\right]^+, \\[2ex] R_{n\rightrightarrows} &= F, \end{cases} \tag{22}$$

[20] If paying ℓ represents a physical cost, then $(1-\theta)y^* = \ell$, i.e. the cost of liquidation equals the cost of continuation. If it includes a transfer to unmodeled creditors (like AOT's "outside obligation" v), then $y^* < \ell$. Working with a general y^* allows us to stay agnostic on the interpretation of ℓ.

with shocked banks being liquidated whenever $F < \frac{N-1}{N-S}(\ell - \theta y)$. Thus the efficient outcome is implemented—banks are liquidated if $y < y^*$ but not if $y \geq y^*$—by setting $F = \frac{N-1}{N-S}(\ell - \theta y^*)$.

Intuitively, no matter the value of y, high debts allow banks to raise liquidity. In the baseline, that is only a good thing. Here, it can be bad. But an appropriately chosen debt level implements the efficient outcome no matter the value of y.

6.2 Costly Default

So far, we assumed that, while liquidation entailed a deadweight loss, default was just a transfer. Thus we found that higher debts always (weakly) increased efficiency (in the sense of Proposition 2). An alternative notion of efficiency could minimize defaults as well as liquidations. Here we show that an exponential network can achieve this goal if the debt levels are not too high, albeit only in an example.

Here we return to the three-bank exponential network in the example in Section 5 and replace the network $\mathbf{F}$ in equation (18) with $\alpha\mathbf{F}$, so increasing α increases indebtedness. Recall that the parameters are such that with two shocked banks the constrained efficient number of liquidations is one. As shocked banks always default (see equation (4)), the constrained efficient number of defaults is two. In the example in Section 5, we achieve the efficient number of liquidations, but not of defaults (all three banks default). Here we do both.

We achieve both goals by reducing the debt. To see that, set $\alpha = 5/8$ and observe that the clearing vector equilibrium payment is $\mathbf{R}_{\rightrightarrows} = (\frac{1}{40}, 0, \frac{15}{16})$: B_1 is not liquidated and B_3 does not default. (But reducing the debt too much undermines the flexibility to avoid liquidation. If debt is too low, say $\alpha = 1/2$, then the clearing vector is $\mathbf{R}_{\rightrightarrows} = (0, 0, \frac{3}{4})$: Both B_1 and B_2 are liquidated.)

Overall, debt should be high enough to allow banks to raise liquidity, but, as in Section 6.1, it should not be too high, this time because that would lead to unnecessary costly defaults.

6.3 Heterogeneous Banks

So far, we assumed that all banks had the same assets in place and the same size liquidity shocks. Now we allow them to be heterogeneous, with y_i denoting B_i's assets and ℓ_i the size of its potential liquidity shock. We denote the deadweight loss of liquidating B_i by $\Delta_i \geq 0$ (with the interpretation described in footnote 20).

We start with a generalization of the planner's problem to this environment:

Definition 11 (Generalized planner's problem (PP)). *The* (generalized) planner's problem *is to find a vector of transfers* $\mathbf{t}$ *for each* $\boldsymbol{\sigma}$*, with* t_i *paid to each* B_i*, to minimize the total deadweight loss of liquidation,*

$$\text{minimize} \quad \sum_{i=1}^{N} \mathbb{1}_{\{\theta y_i + t_i < \ell_i \sigma_i\}} \Delta_i, \tag{23}$$

subject to (i) each bank's liquidity constraint

$$t_i \geq \min\{\ell_i \sigma_i - \theta y_i, 0\} \tag{24}$$

and (ii) total liquidity being conserved

$$\sum_{i=1}^{N} t_i \leq 0. \tag{25}$$

Observe that if Δ_i is the same for all banks, then the objective is just to minimize the number of liquidations. If $\ell_i - \theta y_i$ is also the same, the constraints are too and the problem coincides with the planner's problem in Definition 7.

We next show that the problem of choosing transfers among banks is equivalent to that of choosing the set of banks to liquidate, with effectively the same objective and subject to the same constraint on aggregate liquidity:

Definition 12 (Knapsack problem (KP)). *Find a vector of binary variables* $\mathbf{x} \in \{0,1\}^N$ *for* *each $\boldsymbol{\sigma}$, with $x_i = 0$ if B_i is liquidated, to minimize the deadweight loss,*

$$\text{minimize} \quad \sum_{i=1}^{N}(1 - x_i)\Delta_i, \tag{26}$$

subject to liquidity being conserved

$$\sum_{i=1}^{N} x_i\sigma_i(\ell_i - \theta y_i) \leq \sum_{i=1}^{N}(1 - \sigma_i)\theta y_i. \tag{27}$$

In computer science, this problem is called the "knapsack problem," as it describes a problem of finding the most valuable set of objects to put in a knapsack subject to a constraint on their total weight, just as ours describes finding the most valuable set of banks to save subject to a constraint on the total liquidity required, an equivalence formalized in the next result:

Proposition 8 (The planner's problem is the knapsack problem)**.** *PP in Definition 11 is equivalent to KP in Definition 12 in that (i) if $\hat{\mathbf{t}}$ solves PP then $\hat{\mathbf{x}}$ with $\hat{x}_i := \mathbb{1}_{\{\theta y_i + \hat{t}_i \geq \ell_i \sigma_i\}}$ solves KP and (ii) if $\check{\mathbf{x}}$ solves KP then $\check{\mathbf{t}}$ with $\check{t}_i := \sigma_i \check{x}_i(\ell_i - \theta y_i) - (1 - \sigma_i)\theta y_i$ solves PP.*

The knapsack problem is hard to solve (it is NP-hard, in the language of computer science). Hence algorithms have been developed that deliver approximate solutions quickly. One is the "greedy algorithm," defined in our context as follows:

Definition 13 (Greedy algorithm)**.** *Suppose banks are ranked by a permutation π^{-1} on $\{1, ..., N\}$, so $\pi(r)$ is the index of the r-th highest ranked bank. For each $B_{\pi(r)}$, set $x_{\pi(r)} = 1$ if either $\sigma_{\pi(r)} = 0$ or if*

$$\sum_{r'=1}^{r} \sigma_{\pi(r')}\big(\ell_{\pi(r')} - \theta y_{\pi(r')}\big) \leq \sum_{r'=1}^{N} \big(1 - \sigma_{\pi(r')}\big)\theta y_{\pi(r')}. \tag{28}$$

In words, for any ranking of banks, the greedy algorithm goes through them sequentially, saving the highest-ranked shocked banks until liquidity runs out (and saving all not-shocked banks as well).

We now show that an exponential network can implement the greedy algorithm:

Proposition 9 (Exponential networks implement the greedy algorithm)**.** *Let banks be ordered by the ranking π in Definition 13: $i = \pi(i)$. There exists a threshold s^* such that the exponential network $\alpha\mathbf{F}$ with base $s < s^*$ implements the outcome obtained by the greedy algorithm when α is sufficiently large.*

This result embodies the idea that the exponential network, like the greedy algorithm, prioritizes what banks should be saved. The result does not depend on the priority ranking given by π.

But the efficiency of the outcome does depend on the priority ranking. The next definition helps define a useful one:

Definition 14 (Profitability index)**.** *The* profitability index *of a bank B_i is the ratio of the benefit to the cost of avoiding liquidation when it is shocked:*

$$\mathrm{PI}_i := \frac{\Delta_i}{\ell_i - \theta y_i}. \tag{29}$$

The profitablity index here is akin to the eponymous ratio in capital budgeting, namely the ratio of the payoff to costs. The greedy algorithm corresponds to the rule prescribed for investment under capital constraints and mutually exclusive projects: Undertake those with the highest profitability indices.

When banks are ranked by their profitability indices, the greedy algorithm, and hence an exponential network, is optimal in some circumstances and nearly optimal in many others:

Corollary 2 (Optimality of exponential network with common costs)**.** *Suppose all shocked banks have the same liquidity shortfall: $\ell_i - \theta y_i = \ell_j - \theta y_j$ for all i and j and let banks be ordered by their profitability indices, $PI_i \geq PI_j$ for $i \leq j$. An exponential network solves the planner's problem in Definition 11.*

Given Proposition 9, the result says that if the cost of saving every bank is the same, then saving those with the highest benefit delivers the optimum.

The algorithm need not be optimal, but it is nearly optimal if the liquidated banks are small:

Corollary 3 (Approximate optimality of exponential network with small banks)**.** *As in Lemma 8, let i^* be the index of the first liquidated bank. An exponential network can deliver a deadweight loss within Δ_{i^*} of the solution to the planner's problem in Definition 11.*

This result, which follows from an application of linear programming to the knapsack problem in Dantzig (1957), implies that the relative inefficiency of the greedy algorithm is small if banks are small, in particular if the deadweight loss of liquidating a single bank, namely B_{i^*}, is small. The result also has an analogy in capital budgeting heuristics: If you use your entire budget, then doing those investments with highest profitability is optimal. In general, the risk is that, due to indivisibility, doing those investments might leave some of your budget unused. That risk is small if investments are small. Although the greedy algorithm could still be far from optimal when Δ_{i^*} is large, limiting results elsewhere in that literature suggest that the greedy algorithm is close to optimal on average (see Calvin and Leung (2003)).

Overall, the results here add support to our finding that exponential networks implement a robust policy. The (approximate) efficiency of the priority rule is not specific to the case in which all banks have the same size of assets in place or liquidity shocks.

7 Conclusion

We revisit systemic risk in financial networks, allowing interbank debts to be long-term, as they often are in practice. We show that the right network of long-term debts provides insurance and can even implement the optimal contingent transfers. We thereby extend the idea that debt embeds contingencies via the option to default by showing that it embeds another option—the option to dilute—and that arranging debts in a network can induce its optimal exercise, even leading to the constrained-efficient outcome in fairly general circumstances. Dilution can substitute for policies that mitigate systemic risk by imposing losses on creditors (e.g., Bernard, Capponi, and Stiglitz (2021)): It implements a "backdoor bail-in," allowing debtors to avoid insolvency by shifting the cost distress onto diluted creditors.

A Proofs

A.1 Proof of Lemma 1 (Isomorphism between benchmark and AOT)

In AOT, the clearing vector is completely described by (Lemma B2, equation (B3)):

$$\boldsymbol{x} = \Big[\min\left\{\, y\mathbf{1}\,,\, \mathbf{Q}\boldsymbol{x} + \boldsymbol{e} + \zeta A\mathbf{1}\,\right\}\Big]^{+}. \tag{30}$$

In the case in which long-term assets are fully destroyed in default, the $\zeta = 0$ case, the equation can be re-written as

$$\boldsymbol{x} = \left[\min\left\{\, y\mathbf{1}\,,\, \mathbf{Q}\boldsymbol{x} + \frac{a-v}{a-v+A}(a-v+A)\mathbf{1} - \epsilon\boldsymbol{\sigma}\,\right\}\right]^{+}, \tag{31}$$

having used that, by definition, $\boldsymbol{e} = \boldsymbol{a} - \boldsymbol{v} - \epsilon\boldsymbol{\sigma}$ (with no cash, AOT's $c_i \equiv 0$) and denoting the profile of shock indicator, a notation AOT do not use, by $\boldsymbol{\sigma}$, as in our baseline. This is equivalent to the the equilibrium in our short-term debt benchmark, which per Definition 1 is given by

$$\mathbf{R}_{\rightrightarrows} = \Big[\min\left\{\, \mathbf{F}_{\rightrightarrows}\,,\, \hat{\mathbf{F}}^{\top}\mathbf{R}_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\Big]^{+}, \tag{32}$$

where the color coding represents the mapping between the notation in the two papers, as described in Table 1.

In both models, the banks default whenever they cannot repay the face value of their debts; hence the sets of defaulting banks coincide. Likewise, in both, all defaulting banks are liquidated; hence efficiency coincides too (see Definition 2). □

A.2 Proof of Lemma 2 (Netting in short-term benchmark)

This proof generalizes AOT's proof of their Proposition 3. The idea is to show that in the equilibrium of $\alpha\mathbf{F}$ for $\alpha > 1$, each bank's shortfall $\mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$ is greater than it is in the

Table 1: Notations in AOT and here.

	AOT	This paper
Face value of debt	y_{ji}	$F_{i\to j}$
Payment received	$[\mathbf{Q}\boldsymbol{x}]_i$	$[\hat{\mathbf{F}}^\top \mathbf{R}_{\rightrightarrows}]_i$
Negative shock	$a - v - e_i$	$\ell\sigma_i$
Total assets	$a - v + A$	y
Pledgeable assets	$a - v$	θy
Non-pledgeable assets	A	$(1-\theta)y$

equilibrium of $\mathbf{F}$ and therefore so is the number of defaults.

Lemma A.1. *Define the mapping*

$$\Psi^\alpha : \mathbf{D} \mapsto \Big[\min\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \hat{\mathbf{F}}^\top\mathbf{D} - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\}\Big]^+. \tag{33}$$

If $\mathbf{R}^\alpha_{\rightrightarrows}$ is a clearing vector of $\alpha\mathbf{F}$, then the "shortfall" $\mathbf{D}^\alpha := \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows}$ is a fixed point of Ψ^α.

Proof. We compute, using $\mathbf{Q} \equiv \hat{\mathbf{F}}^\top$:

$$\begin{aligned}
\alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows} &= \alpha\mathbf{F}_{\rightrightarrows} - \max\big\{\mathbf{0}, \min\{\alpha\mathbf{F}_{\rightrightarrows}, \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\}\big\} && \text{(Payment Eqm.)}\\
&= \min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \max\{\mathbf{0}, \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\}\big\} && \text{(Combining)}\\
&= \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \alpha\mathbf{F}_{\rightrightarrows} - \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ && \text{(Interchange min/max)}\\
&= \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}(\alpha\mathbf{F}_{\rightrightarrows} - \mathbf{R}^\alpha_{\rightrightarrows}) - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ && \text{(zero-net debt)}
\end{aligned}$$

Substituting from the definitions of $\mathbf{D}^\alpha$ and Ψ^α gives the result. □

Now we show that for $\alpha > 1$, any fixed point of Ψ^α is greater than $\mathbf{D}^1$—i.e. that increasing debt levels increases default:

Lemma A.2. *Let $\mathbf{D}^1$ be a fixed point of Ψ^1 and define*

$$\mathscr{H}^\alpha := \prod_{i=1}^{N} \big[D^1_i, \alpha F_{i\rightrightarrows}\big] \tag{34}$$

For $\alpha > 1$, Ψ^α maps $\mathscr{H}^\alpha$ into itself, i.e. $\Psi^\alpha\big(\mathscr{H}^\alpha\big) \subset \mathscr{H}^\alpha$.

Proof. The upper bound, i.e. that $\Psi^\alpha(\mathbf{D}^\alpha) \leq \alpha\mathbf{F}_{\rightrightarrows}$, follows immediately from the definition of Ψ^α as a minimum.

So we need only to show the lower bound, i.e. that $\Psi^\alpha(\mathbf{D}^\alpha) \geq \mathbf{D}^1$ for all $\mathbf{D}^\alpha \in \mathscr{H}^\alpha$. We have that for $\mathbf{D}^\alpha \in \mathscr{H}^\alpha$, $\mathbf{D}^1 \leq \mathbf{D}^\alpha$ by definition of the domain. Thus we can compute:

$$\Psi^\alpha(\mathbf{D}^\alpha) = \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{D}^\alpha - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ \tag{35}$$

$$\geq \Big[\min\big\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{D}^1 - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\big\}\Big]^+ \tag{36}$$

$$\geq \Big[\min\{\mathbf{F}_{\rightrightarrows}, \mathbf{Q}\mathbf{D}^1 - \theta y\mathbf{1} + \ell\boldsymbol{\sigma}\}\Big]^+ \tag{37}$$

$$\equiv \Psi(\mathbf{D}^1) \equiv \mathbf{D}^1, \tag{38}$$

since $\mathbf{D}^1$ is a fixed point of Ψ^1 by definition. □

Combining the two lemmata above and applying Brouwer's theorem, we have that for $\alpha > 1$, an equilibrium of $\alpha\mathbf{F}$ is a fixed point of a mapping on $\mathscr{H}^\alpha$. Therefore the generically unique[21] clearing vector, being in $\mathscr{H}^\alpha$, exceeds $\mathbf{D}^1$. □

A.3 Proof of Lemma 3 (Delta connectedness in short-term benchmark)

This proof mirrors AOT's proof of their Proposition 6; we translate it to our notation, adapt it to our notion of efficiency, and add some details.

Throughout we write $F := F_{i\rightrightarrows}$, w.l.o.g., given the network is regular by assumption.

Proof of statement (i). We show, by verification, that in equilibrium all banks are liquidated and, therefore, no network is less efficient than the ring. Assuming all banks are liquidated and letting the single shocked bank be B_1, w.l.o.g., the equilibrium is the solution

[21] Generic global uniqueness follows from the isomorphism to AOT (Lemma 1) and the generic uniqueness of their equilibrium.

to

$$\begin{cases} R_{1\to 2} &= \left[\theta y - \ell + R_{N\to 1}\right]^+ \\ R_{i\to i+1} &= \theta y + R_{i-1\to i} \qquad\qquad i \in \{2, ..., N\}, \end{cases} \tag{39}$$

with the convention that $N+1 := 1$ for indices. Solving, we have the equilibrium: $R_{1\rightrightarrows} = 0$ and, for $i > 1$, $R_{i\to i+1} = (i-1)\theta y \leq (N-1)\theta y$, which is less than F by hypothesis. I.e., all banks are liquidated.

Proof of statement (ii). Here we show that if $\mathbf{F}$ is δ-connected and δ is small, then not all banks are liquidated and, therefore, the network is more efficient than the ring (in which they are). To do so, we show two lemmata.

Lemma A.3. *Let $\mathscr{B}$ be a subset of banks (not equal to all banks). Suppose that the single shocked bank is not in $\mathscr{B}$. If $\mathbf{F}$ is δ-connected with $\delta < \frac{\theta y}{NF}$, then*

$$\sum_{i\in\mathscr{B}}\sum_{j\in\mathscr{B}^c} R_{i\to j} < \theta y|\mathscr{B}|. \tag{40}$$

The result says that the total payment from banks in $\mathscr{B}$ to those in $\mathscr{B}^c$ is small when δ small.

Proof. By the definition of delta-connectedness, $F_{i\to j} \leq \delta F$, for all $(i,j) \in \mathscr{B}\times\mathscr{B}^c$. Thus $R_{i\to j} \leq F_{i\to j} \leq \delta F$, for all $i \in \mathscr{B}$ and $j \in \mathscr{B}^c$. Now summing i over $\mathscr{B}$, summing j over $\mathscr{B}^c$, and using $\delta < \frac{\theta y}{NF} < \frac{\theta y}{|\mathscr{B}^c|F}$, gives

$$\sum_{i\in\mathscr{B}}\sum_{j\in\mathscr{B}^c} R_{i\to j} \leq \delta F|\mathscr{B}||\mathscr{B}^c| < \theta y|\mathscr{B}|. \tag{41}$$

□

Lemma A.4. *Maintain the assumptions that the single shocked bank is not in $\mathscr{B}$ and that $\mathbf{F}$ is δ-connected with $\delta < \frac{\theta y}{NF}$. If inequality* (40) *holds, then not all banks in $\mathscr{B}$ are liquidated.*

Proof. Suppose, in anticipation of a contradiction, that all banks in $\mathscr{B}$ are liquidated. Thus for each B_j in $\mathscr{B}$, $R_{j\rightrightarrows} = R_{j\leftleftarrows} + \theta y$. Summing over banks in $\mathscr{B}$, we have that $\sum_{j\in\mathscr{B}} R_{j\rightrightarrows} =$

$\sum_{j\in\mathscr{B}} R_{j\leftleftarrows} + \theta y|\mathscr{B}|$. Now just expand $R_{j\rightrightarrows}$ and $R_{j\leftleftarrows}$ into their component payments,

$$\sum_{j\in\mathscr{B}}\left(\sum_{i\in\mathscr{B}} R_{j\to i} + \sum_{i\in\mathscr{B}^c} R_{j\to i}\right) = \sum_{j\in\mathscr{B}}\left(\sum_{i\in\mathscr{B}} R_{i\to j} + \sum_{i\in\mathscr{B}^c} R_{i\to j}\right) + \theta y|\mathscr{B}|, \tag{42}$$

and cancel $\sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}} R_{j\to i}$ to get that

$$\sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}^c} R_{j\to i} = \sum_{j\in\mathscr{B}}\sum_{i\in\mathscr{B}^c} R_{i\to j} + \theta y|\mathscr{B}| \geq \theta y|\mathscr{B}| \tag{43}$$

This contradicts equation (40), which says that the total payment from $\mathscr{B}$ to $\mathscr{B}^c$ is small.

□

A.4 Proof of Lemma 4 (Default radius in short-term benchmark)

The result is the same as AOT's. Hence we omit the proof. □

A.5 Proof of Lemma 5 (Bottleneck connectedness in short-term benchmark)

Although the result is close to AOT's Proposition 8, the proof is new, as AOT do not include one.

We begin with a lemma that connects the bottleneck parameter β to the harmonic distance d:

Lemma A.5. *Let* $\mathbf{F}$ *be a symmetric financial network of* N *banks. The bottleneck parameter* β *satisfies*

$$\frac{1}{2N\beta} \leq \max_{i,k\,:\,i\neq k} d_{k\to i} \leq \frac{16}{N\beta^2}. \tag{44}$$

Proof. This is AOT's Lemma 1 (p. 580). Hence we omit the proof.

□

With this we proceed to the two statements of the lemma.

Proof of statement (i). For $\beta > \beta^{ST} \equiv 4\sqrt{\frac{\theta y}{NF}}$, we have, from Lemma A.5, that

$$d_{j\to i} \leq \frac{16}{N\beta^2} < \frac{16}{N(\beta^{ST})^2} = \frac{F}{\theta y} \equiv d^{ST} \quad \text{for all} \quad i \neq j, \tag{45}$$

per the definition of d^{ST} in Lemma 4. That lemma implies that when B_j is shocked, then each B_i defaults.

Proof of statement (ii). For $\beta < \beta_{ST} \equiv \min\left\{\frac{\theta y}{2NF}\,,\, 1\right\}$, we have, from Lemma A.5, that, for some i,

$$d_{j\to i} \geq \frac{1}{2N\beta} > \frac{1}{2N\beta_{ST}} = \max\left\{\frac{F}{\theta y}\,,\, \frac{1}{2N}\right\} \equiv \max\left\{d^{ST}\,,\, \frac{1}{2N}\right\} \geq d^{ST}, \tag{46}$$

per the definition of d^{ST} in Lemma 4. That lemma implies that when B_j is shocked, then B_i does not default. □

A.6 Proof of Proposition 1 (Existence and uniqueness)

Per Section 3, the clearing vector satisfies the same equations in the long- and short-term debt network (only the sets of liquidated banks are different). Thus, given Lemma 1, existence of and generic uniqueness of the clearing vector follow from the analogous results in AOT (their Proposition 1, p. 572). □

A.7 Proof of Proposition 2 (Netting)

The proof is similar to that of Lemma 2, but simpler because we work with repayments $\mathbf{R}_{\rightrightarrows}$ directly instead of "shortfalls" $\mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$.

Define the mapping

$$\Phi^{\alpha}(\mathbf{R}) := \left[\min\left\{\alpha\mathbf{F}_{\rightrightarrows}\,,\, \mathbf{Q}\mathbf{R} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^{+}. \tag{47}$$

Keeping in mind that $\mathbf{Q} \equiv \hat{\mathbf{F}}^{\top}$, the fixed point of Φ^1 is a clearing vector of $\mathbf{F}$. □

Lemma A.6. *Let* $\mathbf{R}^1_{\rightrightarrows}$ *be a fixed point of* Φ^1 *and define*

$$\mathscr{I}^\alpha := \prod_{i=1}^{N} \left[R^1_{i\rightrightarrows}, \alpha F_{i\rightrightarrows}\right]. \tag{48}$$

For $\alpha > 1$, Φ^α *maps* $\mathscr{I}^\alpha$ *into itself, i.e. that* $\Phi^\alpha(\mathscr{I}^\alpha) \subset \mathscr{I}^\alpha$.

Proof. The upper bound, i.e. that $\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) \leq \alpha\mathbf{F}_{\rightrightarrows}$, follows immediately from the definition of Φ^α as a minimum.

So we need only to show the lower bound, i.e. that $\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) \geq \mathbf{R}^1_{\rightrightarrows}$. For $\mathbf{R}^\alpha_{\rightrightarrows} \in \mathscr{I}^\alpha$, we can compute:

$$\begin{aligned}
\Phi^\alpha(\mathbf{R}^\alpha_{\rightrightarrows}) &= \left[\min\left\{\alpha\mathbf{F}_{\rightrightarrows}\,,\ \mathbf{Q}\mathbf{R}^\alpha_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^+ &(49)\\
&\geq \left[\min\left\{\alpha\mathbf{F}_{\rightrightarrows}\,,\ \mathbf{Q}\mathbf{R}^1_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^+ &(50)\\
&\geq \left[\min\left\{\mathbf{F}_{\rightrightarrows}\,,\ \mathbf{Q}\mathbf{R}^1_{\rightrightarrows} + \theta y\mathbf{1} - \ell\boldsymbol{\sigma}\right\}\right]^+ &(51)\\
&\equiv \Phi^1(\mathbf{R}^1_{\rightrightarrows}) \equiv \mathbf{R}^1_{\rightrightarrows}, &(52)
\end{aligned}$$

since $\mathbf{R}^1_{\rightrightarrows}$ is a fixed point of Φ by definition.

□

Given the lemma, we can apply Brouwer's theorem, to conclude that for $\alpha > 1$, an equilibrium of $\alpha\mathbf{F}$ is a fixed point of a mapping on $\mathscr{I}^\alpha$. Therefore the generically unique[22] clearing vector, being in $\mathscr{I}^\alpha$, exceeds $\mathbf{R}^1_{\rightrightarrows}$: All repayments are higher when α is higher, so there are fewer liquidations. □

A.8 Proof of Proposition 3 (Delta connectedness)

Note that this proof makes use of the minimum number of liquidated banks, L^*, derived in Lemma 6, even though that result comes later in the text.

[22] Generic global uniqueness follows from Proposition 1.

Throughout we assume, w.l.o.g., that the B_1 is the not-shocked bank.

Proof of statement (i). Equation (4) and the fact that banks have zero net positions (so no shocked bank can repay in full) imply that any shocked bank ($i \geq 2$) repays

$$R_{i\rightrightarrows} \equiv R_{i\to i+1} = \left[\theta y - \ell + R_{i\leftleftarrows}\right]^{+} \tag{53}$$

$$= \left[\theta y - \ell + R_{i-1\to i}\right]^{+}, \tag{54}$$

having used the definition of the ring network (Definition 3).

The expression for $R_{i\rightrightarrows}$ implies that if B_{i-1} is liquidated, then B_i is too. Thus the number of banks that are not liquidated is the maximum index i for which $R_{i-1\to i} \geq \ell - \theta y$.

We can now expand the condition recursively for any B_i that is not liquidated:

$$\ell - \theta y \leq R_{i-1\to i} = \theta y - \ell + R_{i-2\to i-1} \tag{55}$$

$$= k(\theta y - \ell) + R_{i-(k+1)\to i-k} \quad (\text{for } k \in \{1, ..., i-2\}) \tag{56}$$

$$= (i-2)(\theta y - \ell) + R_{1\to 2}. \tag{57}$$

So the number of banks that are not liquidated is

$$\max\left\{ i \leq N \,:\, i - 1 \leq \frac{R_{1\to 2}}{\ell - \theta y} \right\} = \min\left\{ N\,,\, \left\lfloor 1 + \frac{R_{1\to 2}}{\ell - \theta y} \right\rfloor \right\}. \tag{58}$$

The number of liquidated banks is N minus the above:

$$L = \max\left\{ 0\,,\, \left\lceil N - 1 - \frac{R_{1\to 2}}{\ell - \theta y} \right\rceil \right\}. \tag{59}$$

For $F > \theta y$, per the condition of the proposition, $R_{1\to 2} = \theta y$ and the minimum number of liquidations is attained $L = L^*$ (equation (10) with $S = N - 1$ and ℓ sufficiently large, per the statement of the result).

Statement (ii). A trivial example suffices: The "linkless" network—$F_{i\to j} = 0$ for all i

and j—is δ-connected for any δ; in it, all shocked banks are liquidated. That is less efficient than the ring network with $F \geq \theta y$, per statement (i). □

A.9 Proof of Proposition 4 (Salvation radius)

We prove the two statements in turn. The arguments build on AOT's proof of their Proposition 8 (pp. 602–603). Ours are a bit more complicated because we cannot work with the clearing vector $\mathbf{R}_{\rightrightarrows}$, but have to work with the shortfall $\mathbf{D} \equiv \mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$ instead.

Statement (i). The proof comprises (somewhat involved) calculations using the shortfall $\mathbf{D}$, which ultimately allow us to bound the harmonic distance of the not-shocked banks to any liquidated bank.

Throughout we denote the set of defaulting banks by $\mathscr{D}$, that of liquidated banks by $\mathscr{L}$, and, hence, that of those that default but are not liquidated by $\mathscr{D} \setminus \mathscr{L}$. $\mathbf{1}$ denotes the vector of all ones and $\mathbf{I}$ the identity matrix, each of appropriate dimension determined by the context. Given the network is regular, we can write $F_{i\rightrightarrows} \equiv F$.

We start with two lemmata. Each takes as its starting point the equilibrium equation for the shortfall

$$\mathbf{D} = \Big[\min\{\mathbf{F}_{\rightrightarrows}, \mathbf{QD} - \theta y \mathbf{1} + \ell \boldsymbol{\sigma}\} \Big]^{+}, \tag{60}$$

which follows from setting $\alpha = 1$ in Lemma A.1. The first lemma develops the equation for banks that default but are not liquidated; the second for banks that are liquidated.

Lemma A.7. $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} = \big(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\big)^{-1}\big(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{1}F - (\theta y - \ell)\mathbf{1}\big).$

Proof. For banks that default, the shortfall is not zero, and for those that are not liquidated, it is less than F. Hence for B_i in $\mathscr{D}\setminus\mathscr{L}$, the second term under the "min" in equation (60) is the relevant one; writing that elementwise gives the following:

$$D_i = \sum_{k=1}^{N} Q_{ik} D_k - \theta y + \ell \tag{61}$$

$$= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k + \sum_{k\notin\mathscr{D}} Q_{ik} D_k - \theta y + \ell \tag{62}$$

$$= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k - \theta y + \ell, \tag{63}$$

having used that shortfall is zero for banks that do not default ($D_k = 0$ for $k \notin \mathscr{D}$). Rewriting the above in block matrix form, using that fact that liquidated banks repay nothing ($D_k = F$ for $k \in \mathscr{L}$), and rearranging gives:

$$\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} = \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}} \mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} \mathbf{D}_{\mathscr{L}} - (\theta y - \ell)\mathbf{1} \tag{64}$$

$$= \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}} \mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} F\mathbf{1} - (\theta y - \ell)\mathbf{1} \tag{65}$$

$$= (\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} F\mathbf{1} - (\theta y - \ell)\mathbf{1}), \tag{66}$$

where the last expression comes from solving for $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}}$ and rearranging. □

Lemma A.8. $\big(\mathbf{I} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}\big)\big(\ell - \theta y\big)\mathbf{1} > \tilde{\mathbf{Q}} F\mathbf{1}$, *where*

$$\tilde{\mathbf{Q}} := \big(\mathbf{I} - \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} - \mathbf{Q}_{\mathscr{L},\mathscr{L}}\big). \tag{67}$$

Proof. Banks that are liquidated repay zero (equation (4)), so

$$\begin{aligned} F &< \sum_{k=1}^{N} Q_{ik} D_k - \theta y + \ell && (68) \\ &= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k + \sum_{k\notin\mathscr{D}} Q_{ik} D_k - \theta y + \ell && (69) \\ &= \sum_{k\in\mathscr{L}} Q_{ik} D_k + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k - \theta y + \ell && (70) \\ &= \sum_{k\in\mathscr{L}} Q_{ik} F + \sum_{k\in\mathscr{D}\setminus\mathscr{L}} Q_{ik} D_k - \theta y + \ell, && (71) \end{aligned}$$

having used that shortfall is zero for banks that do not default ($D_k = 0$ for $k \notin \mathscr{D}$) and F for those that are liquidated ($D_k = F$ for $k \in \mathscr{L}$). The above can be re-written in block-matrix notation, $F\mathbf{1} < \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1}$, so the expression for $\mathbf{D}_{\mathscr{D}\setminus\mathscr{L}}$ from Lemma A.7 can be substituted in to get:

$$\mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1}) + \mathbf{Q}_{\mathscr{L},\mathscr{L}}F\mathbf{1} - (\theta y - \ell)\mathbf{1} > F\mathbf{1}. \quad (72)$$

Rearranging the above gives the expression in the lemma. □

Now we compute a bound on the harmonic distance d. First we use the definition of d (Definition 5) to write in block matrix form:

$$\begin{cases} \mathbf{d}_{j\to\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}, \\ \mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}}\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}, \end{cases} \quad (73)$$

where $\mathbf{d}_{j\to\mathscr{L}}$ and $\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}$ are vectors that capture the harmonic distances from B_j to each of (i) the liquidated and (ii) the defaulting but not liquidated banks, respectively (cf. equations (B19) and (B20) in AOT). (NB: As, by hypothesis, B_j is the only not-shocked bank, there are no additional terms to not defaulting banks.)

Solving for the system in equation (73)—solving for $\mathbf{d}_{j\to\mathscr{D}\setminus\mathscr{L}}$ in the second equation and substituting it into the first—gives

$$\mathbf{d}_{j\to\mathscr{L}} = \mathbf{1} + \mathbf{Q}_{\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1}(\mathbf{1} + \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}}\mathbf{d}_{j\to\mathscr{L}}) \tag{74}$$

or, given the definition of $\tilde{\mathbf{Q}}$ in equation (67), $\tilde{\mathbf{Q}}\mathbf{d}_{j\to\mathscr{L}} = (\mathbf{I} + \mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}}(\mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}})^{-1})\mathbf{1}$. From here, we can use Lemma A.8 to write

$$\tilde{\mathbf{Q}}\mathbf{d}_{j\to\mathscr{L}} > \tilde{\mathbf{Q}}\frac{F}{\ell - \theta y}\mathbf{1}. \tag{75}$$

As $\tilde{\mathbf{Q}}$ is invertibe and elementwise non-negative,[23] this says that if B_i is liquidated, then

$$d_{j\to i} \geq \frac{F}{\ell - \theta y} \equiv d^{LT}, \tag{76}$$

per the definition of d^{LT} in the proposition. That is the desired result.

Statement (ii). As, by hypothesis, no bank is liquidated, we have that for any B_i $D_i < F_{i\rightrightarrows}$, from the definition of the shortfall $\mathbf{D}$. Thus, from the equilibrium equation for the shortfall (Lemma A.1 with $\alpha = 1$) and the observation that no shocked bank repays in full, $D_i > 0$ (equation (4) given the assumption that banks have zero net positions), we have

$$D_i = \sum_{k\neq i} Q_{ik}D_k + \ell - \theta y, \tag{77}$$

for any B_i for $i \neq j$, where, remember, B_j is the not-shocked bank. Dividing both sides of equation (77) by $\ell - \theta y$ says that $D_i/(\ell - \theta y)$ solves $x_i = 1 + \sum_{k\neq i} Q_{ik}x_k$ for all i. By the

[23]The result follows Theorem 2 of Plemmons (1977) and exercise 5.8 of Berman and Plemmons (1979, p. 159) given that $\tilde{\mathbf{Q}}$ is the Schur complement of the non-singular M-matrix

$$\begin{bmatrix} \mathbf{I} - \mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{D}\setminus\mathscr{L}} & -\mathbf{Q}_{\mathscr{D}\setminus\mathscr{L},\mathscr{L}} \\ -\mathbf{Q}_{\mathscr{L},\mathscr{D}\setminus\mathscr{L}} & \mathbf{I} - \mathbf{Q}_{\mathscr{L},\mathscr{L}} \end{bmatrix}.$$

definition (and uniqueness) of the harmonic distance, that implies that

$$d_{j\to i} = \frac{D_i}{\ell - \theta y}. \tag{78}$$

As $D_i < F$ by hypothesis, $d_{j\to i} < F/(\ell - \theta y) \equiv d^{LT}$, as desired. □

A.10 Proof of Proposition 5 (Bottleneck connectedness)

We prove the two statements of the lemma sequentially. They rely on Lemma A.5 above (which, recall, depends only on the network structure, not the maturity of debt despite being stated within the short-term debt benchmark).

Proof of statement (i). For $\beta > \beta^{LT} \equiv 4\sqrt{\frac{\ell-\theta y}{NF}}$, we have, from Lemma A.5, that

$$d_{j\to i} \leq \frac{16}{N\beta^2} < \frac{16}{N(\beta^{LT})^2} = \frac{F}{\ell - \theta y} \equiv d^{LT}, \tag{79}$$

per the definition of d^{LT} in Proposition 4. That result implies that when B_j is not shocked, then B_i is not liquidated.

Proof of statement (ii). For $\beta < \beta_{LT} \equiv \min\left\{\frac{\ell-\theta y}{2NF}, 1\right\}$, we have, from Lemma A.5, that, for some i,

$$d_{j\to i} \geq \frac{1}{2N\beta} > \frac{1}{2N\beta_{LT}} = \max\left\{\frac{F}{\ell-\theta y}, \frac{1}{2N}\right\} \equiv \max\left\{d^{LT}, \frac{1}{2N}\right\} \geq d^{LT}, \tag{80}$$

per the definition of d^{LT} in Proposition 4. That result implies that when B_j is not shocked, then B_i does not default and, hence, not all banks are liquidated. □

A.11 Proof of Lemma 6 (Constrained efficiency)

The argument is in the text. □

A.12 Proof of Lemma 7 (High debt mutualizes assets)

To prove the result, we show that if a not-shocked bank, say B_i, makes net payment less than θy no matter how high α is, it is impossible that another bank, say B_j, is liquidated. From equation (5), the not-shocked bank's net payment is

$$R_{i\rightrightarrows} - R_{i\leftleftarrows} = \begin{cases} \theta y & \text{if defaults,} \\ \alpha F_{i\rightrightarrows} - R_{i\leftleftarrows} & \text{otherwise.} \end{cases} \tag{81}$$

Now suppose (in anticipation of a contradiction) that B_i's payment is strictly less than θy and another bank, say B_j, is liquidated. From equation (2), that implies that

$$\begin{aligned} \ell &> \theta y + R_{j\leftleftarrows} && (82)\\ &= \theta y + \sum_{k \neq i,j} R_{k\to j} + R_{i\to j} && (83)\\ &= \theta y + \sum_{k \neq i,j} R_{k\to j} + \alpha F_{i\to j}, && (84) \end{aligned}$$

having used the fact that B_i repays B_j in full (otherwise its net payment would equal θy). The inequality cannot hold for large α (as $F_{i\to j} > 0$ by the assumption that the network is connected). Thus either no bank is liquidated or B_i's net payment cannot be less than θy. □

A.13 Proof of Corollary 1 (First best with high debt for small shocks)

Suppose (in anticipation of a contradiction) that at least one bank is liquidated. From Lemma 7 and equation (4), we know that for each B_i the net payment is

$$R_{i\rightrightarrows} - R_{i\leftleftarrows} \begin{cases} = \theta y - \ell\sigma_i & \text{if } B_i \text{ is not liquidated,} \\ > \theta y - \ell\sigma_i & \text{if } B_i \text{ is liquidated.} \end{cases} \tag{85}$$

Combining market clearing (equation (6)) with the expression above, we have that

$$0 = \sum \left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) > \sum \left(\theta y - \ell\sigma_i\right) = N\theta y - S\ell, \tag{86}$$

contradicting the hypothesis that $N\theta y > S\ell$. Therefore no bank can be liquidated, as desired. □

A.14 Proof of Lemma 8 (Controlling relative payments to liquidated banks)

Here we use the pro rata condition that $R_{i\to j} = \hat{F}_{i\to j} R_{i\rightrightarrows}$ for any j (including $j = i^*$) to write

$$R_{i\to j} = \frac{\hat{F}_{i\to j}}{\hat{F}_{i\to i^*}} R_{i\to i^*}. \tag{87}$$

Thus the total payment to any liquidated bank B_j is

$$R_{j\leftleftarrows} = \sum_{i\neq j,i^*} R_{i\to j} + R_{i^*\to j} \tag{88}$$

$$= \sum_{i\neq j,i^*} \frac{\hat{F}_{i\to j}}{\hat{F}_{i\to i^*}} R_{i\to i^*}, \tag{89}$$

having used that $R_{i^*\to j} = 0$ as any liquidated bank makes zero repayment (equation (4)). Building on the above by substituting from the definition of s-dominance and adding the non-negative term $s^{j-i^*}R_{j\to i^*}$ gives

$$R_{j\leftleftarrows} \leq \sum_{i\neq j,i^*} s^{j-i^*}R_{i\to i^*} + s^{j-i^*}R_{j\to i^*} = s^{j-i^*}R_{i^*\leftleftarrows}. \tag{90}$$

□

A.15 Proof of Lemma 9 (Controlling total payments to liquidated banks)

Since liquidated banks make zero repayments, $R_{i\rightrightarrows} = 0$ for $i \in \mathscr{L}$, equation (4) implies that each receives payment $R_{i\leftleftarrows} < \ell - \theta y$. Applying this to B_{i^*}, the largest liquidated bank, and using Lemma 8, we have the following:

$$\sum_{i\in\mathscr{L}} R_{i\leftleftarrows} \leq \sum_{i\in\mathscr{L}} s^{i-i^*}R_{i^*\leftleftarrows} \tag{91}$$

$$\leq R_{i^*\leftleftarrows}\sum_{i=0}^{\infty} s^i \tag{92}$$

$$= R_{i^*\leftleftarrows}\frac{1}{1-s} \tag{93}$$

$$< \frac{\ell-\theta y}{1-s}. \tag{94}$$

□

A.16 Proof of Proposition 6 (Efficiency of exponential networks)

As the exponential network is connected, we know from Lemma 6 and Corollary 1 that if $L^* = 0$ then no bank is liquidated as long as α is high. Hence we focus on the case in which at least one bank is liquidated in the constrained-efficient outcome, $L^* \geq 1$.

Recall that in this case it suffices to show the following:

(i) Each bank that is not liquidated makes the maximum net payment it can without being liquidated, $R_{i\rightrightarrows} - R_{i\leftleftarrows} = \theta y - \ell\sigma_i$, for B_i not liquidated.

(ii) The banks that are liquidated receive a total net payment that would be insufficient to save any one of them, $-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) < \ell - \theta y$.

The first property follows from Lemma 7 and equation (4).

The second property follows from two steps. The first is to use Lemma 9 and the definition of s^* to bound the liquidated banks' net payment in terms of L^*:

$$-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) < \frac{\ell-\theta y}{1-s} \tag{95}$$

$$\leq \frac{\ell-\theta y}{1-s^*} \tag{96}$$

$$= N\theta y - S\ell + (1+L^*)(\ell-\theta y). \tag{97}$$

The second step is to use market clearing, $\sum_{i=1}^{N}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) = 0$ by equation (6), to write the LHS above in terms of the number of liquidated banks L, using that (i) not-shocked banks make net payment θy (by Lemma 7) and (ii) shocked, not-liquidated banks make net payment $\theta y - \ell$ (by equation (4)):

$$-\sum_{i\in\mathscr{L}}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) = \sum_{i\in\mathscr{L}^c}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) \tag{98}$$

$$= \sum_{i\in\mathscr{L}^c:\sigma_i=0}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) + \sum_{i\in\mathscr{L}^c:\sigma_i=1}\left(R_{i\rightrightarrows} - R_{i\leftleftarrows}\right) \tag{99}$$

$$= (N-S)\theta y + (S-L)(\theta y - \ell). \tag{100}$$

Combining this with the bound in equation (97) and canceling terms says $L < 1 + L^*$. As L and L^* are integers, and $L^* \leq L$ by Lemma 6, it must be that $L = L^*$.

(The assumption that $\frac{S\ell - N\theta y}{\ell - \theta y}$ not be an integer was required for for $s^* > 0$ and thus for the exponential network with base $s < s^*$ to be well defined.) □

A.17 Proof of Proposition 7 (Approximate efficiency of exponential networks)

With the weaker notation of efficiency, we need to show only that the banks that are liquidated receive a total net payment insufficient to save any *two* of them, $-\sum_{i \in \mathscr{L}} \big(R_{i\rightrightarrows} - R_{i\leftleftarrows}\big) < 2\big(\ell - \theta y\big)$. Given the proof of Proposition 6, that is all we need to show. As in that proof, it follows Lemma 9 along with the definitions of $s^*(= 1/2)$ and L^*:

$$-\sum_{i \in \mathscr{L}} \big(R_{i\rightrightarrows} - R_{i\leftleftarrows}\big) < \frac{\ell - \theta y}{1 - s^*} \tag{101}$$

$$= 2(\ell - \theta y) \tag{102}$$

as desired. □

A.18 Proof of Lemma 10 (Inefficiency of other networks)

Suppose (in anticipation of a contradiction) that a fully connected network $\mathbf{F}$ achieves constrained efficiency, i.e. that the number of liquidated banks is $L^* = \frac{S\ell - N\theta y}{\ell - \theta y}$, having used the assumptions that $\frac{S\ell - N\theta y}{\ell - \theta y}$ is an integer and that $S\ell > N\theta y$ in conjunction with the definition of L^* (equation (10)).

As each bank that is not shocked pays at most θy and each that is shocked but not liquidated pays exactly $\theta y - \ell\sigma_i$ (equation (4)), we can use market clearing to bound the total payment to the liquidated banks as follows:

$$-\sum_{i \in \mathscr{L}} \big(R_{i\rightrightarrows} - R_{i\leftleftarrows}\big) = \sum_{i \notin \mathscr{L}} \big(R_{i\rightrightarrows} - R_{i\leftleftarrows}\big) \tag{103}$$

$$\leq \sum_{i \notin \mathscr{L}} (\theta y - \sigma_i \ell) \tag{104}$$

$$= (N - S)\theta y + (S - L^*)(\theta y - \ell) \tag{105}$$

$$= 0. \tag{106}$$

I.e. liquidated banks receive no (positive) payment.

But shocked, not-liquidated banks must receive positive payment (otherwise they would be liquidated by equation (4)). Given the hypothesis that the network is fully connected, that contradicts the assumption that payments are pro rata (equation (1)): Any bank that has debt to a not-liquidated bank must have debt to a liquidated bank too and it cannot make a positive payment to one but not the other. □

A.19 Proof of Proposition 8 (The planner's problem is the knapsack problem)

We prove each implication in turn.

(i) $\hat{\mathbf{t}}$ solves PP $\implies$ $\hat{\mathbf{x}}$ solves KP. We show that $\hat{\mathbf{x}}$ is feasible and optimal in turn.

- $\hat{\mathbf{x}}$ *is feasible.* Observe that, since, by equation (24), $\hat{t}_i \geq -\theta y_i$, $\hat{x}_i \sigma_i (\ell_i - \theta y_i) \leq (1 - \sigma_i)\theta y_i + \hat{t}_i$. Thus, using the liquidity conservation constraint (equation (25)), we have that

$$\sum_{i=1}^{N} \hat{x}_i \sigma_i (\ell_i - \theta y_i) \leq \sum_{i=1}^{N} \Big((1 - \sigma_i)\theta y_i + \hat{t}_i \Big) \leq \sum_{i=1}^{N} (1 - \sigma_i)\theta y_i, \tag{107}$$

  confirming that $\hat{\mathbf{x}}$ is feasible.

- $\hat{\mathbf{x}}$ *is optimal.* Suppose, in anticipation of a contradiction, that $\hat{\mathbf{x}}$ is not optimal, i.e. that there is a feasible $\hat{\mathbf{x}}'$ that yields lower deadweight loss (equation (26)).

  We now show that such an $\hat{\mathbf{x}}'$ cannot exist, because, if it does, $\hat{\mathbf{t}}$ cannot be a solution to PP. Specifically, $\hat{\mathbf{t}}'$ with $\hat{t}'_i := \sigma_i \hat{x}'_i (\ell_i - \theta y_i) - (1 - \sigma_i)\theta y_i$ is feasible and yields a lower objective:

  - Feasibility:

    * We show that $\hat{\mathbf{t}}'$ satisfies the liquidity constraint (24) for $\sigma_i = 0$ and $\sigma_i = 1$

in turn: If $\sigma_i = 0$, then $\hat{t}'_i = -\theta y_i = \min\{\sigma_i \ell_i - \theta y_i, 0\}$ and if $\sigma_i = 1$, then $\hat{t}'_i = \hat{x}'_i(\ell_i - \theta y_i) \geq \min\{\sigma_i \ell_i - \theta y_i, 0\}$.

  * We show that $\hat{\mathbf{t}}'$ satisfies the liquidity conservation constraint (25) as

$$\sum_{i=1}^{N} \hat{t}'_i = \sum_{i=1}^{N} \Big(\sigma_i \hat{x}'_i(\ell_i - \theta y_i) - (1 - \sigma_i)\theta y_i\Big) \leq 0, \tag{108}$$

since $\hat{\mathbf{x}}'$ must satisfy the constraint (27) by its definition as a solution to KP.

- Optimality: Observe, from the definitions of $\hat{x}'_i$ and $\hat{t}_i$ that

$$\sum_{i=1}^{N} \mathbb{1}_{\{\theta y_i + \hat{t}'_i < \ell_i \sigma_i\}} \Delta_i \leq \sum_{i=1}^{N} (1 - \hat{x}'_i)\Delta_i < \sum_{i=1}^{N} (1 - \hat{x}_i)\Delta_i = \sum_{i=1}^{N} \mathbb{1}_{\{\theta y_i + \hat{t}_i < \ell_i \sigma_i\}} \Delta_i, \tag{109}$$

contradicting the optimality of $\hat{\mathbf{t}}'$.

Therefore $\hat{\mathbf{x}}$ solves KP.

(ii) $\check{\mathbf{x}}$ solves KP $\implies$ $\check{\mathbf{t}}$ solves PP. We show that $\check{\mathbf{t}}$ is feasible and optimal in turn.

- $\check{\mathbf{t}}$ *is feasible.* Observe that $\check{t}_i$ satisfies B$_i$'s liquidity constraint (24) by construction: $\check{t}_i = \sigma_i \check{x}_i(\ell_i - \theta y_i) - (1 - \sigma_i)\theta y_i \geq \min\{\sigma_i \ell_i - \theta y_i, 0\}$.
- $\check{\mathbf{t}}$ *is optimal.* Suppose, in anticipation of a contradiction, that $\check{\mathbf{t}}$ is not optimal, i.e. that there is a feasible $\check{\mathbf{t}}'$ that yields lower deadweight loss (equation (23)).

  We now show that such an $\check{\mathbf{t}}'$ cannot exist, because, it if does, $\check{\mathbf{x}}$ cannot be a solution to KP. Specifically, $\check{\mathbf{x}}'$ with $\check{x}'_i := \mathbb{1}_{\{\theta y_i + \check{t}'_i - \ell_i \sigma_i \geq 0\}}$ is feasible and yields a lower objective:

  * Feasibility: We show that $\check{x}'_i$ satisfies the liquidity conservation constraint (27) as

$$\sum_{i=1}^{N} \check{x}'_i \sigma_i(\ell_i - \theta y_i) \leq \sum_{i=1}^{N} \Big((1 - \sigma_i)\theta y_i + \check{t}'_i\Big) \leq \sum_{i=1}^{N} (1 - \sigma_i)\theta y_i, \tag{110}$$

since $\check{\mathbf{t}}'$ must satisfy the constraint (25) by its definition as a solution to PP.

* Optimality: Observe, from the definitions of $\check{x}'_i$ and $\check{t}_i$ that

$$\sum_{i=1}^{N}(1-\check{x}'_i)\Delta_i \leq \sum_{i=1}^{N}\mathbb{1}_{\{\theta y_i+\check{t}'_i<\ell_i\sigma_i\}}\Delta_i < \sum_{i=1}^{N}\mathbb{1}_{\{\theta y_i+\check{t}_i<\ell_i\sigma_i\}}\Delta_i \leq \sum_{i=1}^{N}(1-\check{x}_i)\Delta_i, \quad (111)$$

contradicting the optimality of $\check{\mathbf{x}}'$.

Therefore $\check{\mathbf{t}}$ solves PP. □

A.20 Proof of Proposition 9 (Exponential networks implement the greedy algorithm)

First note that Lemma 7, Lemma 8, and Lemma 9 hold with heterogeneous banks. Their proofs are essentially unchanged, with y and ℓ replaced with y_i and ℓ_i everywhere, except in the last line of the proof of Lemma 9, when they are replaced by y_{i^*} and ℓ_{i^*}. We apply these results freely throughout the proof.

Now we prove the result in three steps.

Step 1: **For all i such that $\sigma_i = 0$, $x_i = 1$.** This is immediate from equation (4), which implies that the bank is not liquidated if it is not shocked.

Step 2: **Existence of critical index.** Now show that there exists a critical index i^* such that a shocked bank B_i is liquidated if and only if $i \geq i^*$. Suppose, in anticipation of a contradiction, that, to the contrary, there are shocked B_i and B_j with $i < j$ such that $x_i = 0$ and $x_j = 1$. Then, by equation (4), it must be that $R_{i\leftleftarrows} < \ell_i - \theta y_i$ and $R_{j\leftleftarrows} \geq \ell_j - \theta y_j$ and, therefore, by Lemma 8, that

$$\ell_j - \theta y_j \leq R_{j\leftleftarrows} < s^{j-i}R_{i\leftleftarrows} < s^{j-i}(\ell_i - \theta y_i). \quad (112)$$

The inequality is violated if $s < \sqrt[j-i]{(\ell_j - \theta y_j)/(\ell_i - \theta y_i)}$, a contradiction.

Step 3: **Same critical index.** We now show that the critical index delivered by the exponential network coincides with that delivered by the greedy algorithm (for every state $\boldsymbol{\sigma}$) or, equivalently, that i^* satisfies the following two inequalities:

$$\sum_{i=1}^{i^*} \sigma_i(\ell_i - \theta y_i) > \sum_{i=1}^{N} (1-\sigma_i)\theta y_i \tag{113}$$

and

$$\sum_{i=1}^{i^*-1} \sigma_i(\ell_i - \theta y_i) \leq \sum_{i=1}^{N} (1-\sigma_i)\theta y_i. \tag{114}$$

The second is immediate, as it is implied by the aggregate liquidity constraint, which holds strictly by hypothesis. To prove the first, we invoke Lemma 7, which implies that, as long as debts are sufficiently high, each not-shocked bank pays θy_i in net payment, so conservation of liquidity requires:

$$\sum_{i=1}^{N} (1-\sigma_i)\theta y_i = -\sum_{i\,:\,\sigma_i=0} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) \tag{115}$$

$$= \sum_{i\,:\,\sigma_i=1} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) \tag{116}$$

$$= \sum_{i\in\mathscr{L}^c\,:\,\sigma_i=1} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) + \sum_{i\in\mathscr{L}} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) \tag{117}$$

$$\leq \sum_{i\in\mathscr{L}^c\,:\,\sigma_i=1} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) + \sum_{i\in\mathscr{L}} R_{i\leftleftarrows} \tag{118}$$

$$< \sum_{i\in\mathscr{L}^c\,:\,\sigma_i=1} (R_{i\leftleftarrows} - R_{i\rightrightarrows}) + \frac{R_{i^*\leftleftarrows}}{1-s} \tag{119}$$

$$\leq \sum_{i\in\mathscr{L}^c\,:\,\sigma_i=1} (\ell_i - \theta y_i) + \frac{\ell_{i^*} - \theta y_{i^*}}{1-s} \tag{120}$$

$$= \sum_{i=1}^{i^*} \sigma_i(\ell_i - \theta y_i) + \frac{s}{1-s}(\ell_{i^*} - \theta y^*) \tag{121}$$

having used Lemma 9 to bound the sum over liquidated banks. Letting s be sufficiently small and recalling that the inequality must be strict by assumption gives the result. □

A.21 Proof of Corollary 2 (Optimality of exponential network with common costs)

The result is immediate from the definitions of equivalence of the planner's problem to the knapsack problem and of the greedy algorithm to the exponential network; see Proposition 8, Definition 12, and Proposition 9. (See also the discussion following the statement of the corollary.) □

A.22 Proof of Corollary 3 (Approximate optimality of exponential network with small banks)

The result follows immediately from the so-called Dantzig bound (Dantzig (1957)), which we state as a lemma:

Lemma A.9. *Suppose, w.l.o.g., that banks are ordered by their profitability indices ($PI_i \geq PI_j$ for $i \leq j$) and let $\check{\mathbf{x}}$ be a solution of KP (Definition 12). We have that*

$$\sum_{i=1}^{N} \sigma_i \check{x}_i \Delta_i \leq \sum_{i=1}^{i^*-1} \sigma_i \Delta_i + \Delta_{i^*}. \tag{122}$$

Proof. See Martello and Toth (1990), Theorem 2.1. □

The result implies that the difference between the objective at the optimum (represented by the LHS of equation (122)) and at the greedy alogithm's approximation of it (represented by the sum on the RHS) is at most Δ_{i^*}, as desired. □

B Notations

To the extent possible, we use bold face letters for matrices and vectors and use italics for scalars; we use single-arrow subscripts for liabilities from one bank to another and double-arrow subscripts for total liabilities from one to many banks. E.g., $\mathbf{F} = [F_{i\to j}]_{ij}$ is the matrix of interbank liabilities between individual banks; $\mathbf{F}_{\rightrightarrows} = [F_{i\rightrightarrows}]_i$ is the vector of banks' total interbank liabilities, i.e. the vector of row sums of $\mathbf{F}$. We use B_i for individual banks and script letters for sets; $\mathrm{B}_i \in \mathscr{B}$ and $i \in \mathscr{B}$ are synonymous. We summarize our notations in Table 2, separating those used in the main text from those used only in extensions or proofs.

Table 2: Notations.

Notation	Meaning	Parametric restriction
y	Long-term real asset value	$y > 0$
ℓ	Size of liquidity shock	$\theta y < \ell < y$
θ	Pledgeable fraction of y	$0 < \theta < 1$
σ_i	Indicator of B_i's shock	$\sigma_i \in \{0, 1\}$
$\boldsymbol{\sigma} \equiv \{\sigma_i\}_i$	Vector of shocks/Aggregate state	$\boldsymbol{\sigma} \in \{0, 1\}^N$
B_i	ith bank	
$\mathscr{B}$	A set of banks	
$\mathscr{B}^c$	Complement of $\mathscr{B}$	
$\mathscr{L}$	Set of banks that are liquidated	$\mathscr{L} \subset \mathscr{D}$
N	Number of banks	
$S = \sum \sigma_i$	Number of shocked banks	
$L = \lvert\mathscr{L}\rvert$	Number of liquidated banks	
L^*	Minimum L (Lemma 6)	
$F_{i\to j}$	B_i's liability to B_j	$F_{i\to j} \geq 0$
$\mathbf{F} \equiv [F_{i\to j}]_{ij}$	Matrix of interbank debts	
$F_{i\rightrightarrows} \equiv \sum_{j\neq i} F_{i\to j}$	B_i's total interbank liabilities	
F	Each bank's total liabilities $F \equiv F_{i\rightrightarrows}$ in a regular network	
$F_{i\leftleftarrows} \equiv \sum_{j\neq i} F_{j\to i}$	B_i's total interbank claims	
$\mathbf{F}_{\rightrightarrows} \equiv \{F_{i\rightrightarrows}\}_i$	Vector of each bank's total interbank liabilities	
$\hat{F}_{i\to j} \equiv F_{i\to j}/F_{i\rightrightarrows}$	B_i's liability to B_j as a fraction of its total liabilities	$0 \leq \hat{F}_{i\to j} \leq 1$
$\hat{\mathbf{F}} \equiv [\hat{F}_{i\to j}]_{ij}$	Matrix of interbank debts	$\sum_i \hat{F}_{i\to j} = 1$
$R_{i\to j}$	B_i's equilibrium repayment to B_j	$0 \leq R_{i\to j} \leq F_{i\to j}$
$R_{i\rightrightarrows} \equiv \sum_{j\neq i} R_{i\to j}$	B_i's total repayment to other banks	
$R_{i\leftleftarrows} \equiv \sum_{j\neq i} F_{j\to i}$	B_i's total repayment received from other banks	

Continued on next page

Notations (continued)

Notation	Meaning	Parametric restriction
$\mathbf{R}_{\rightrightarrows} \equiv \{R_{i\rightrightarrows}\}_i$	Vector of each bank's total equilibrium repayment	$\mathbf{0} \leq \mathbf{R}_{\rightrightarrows} \leq \mathbf{F}_{\rightrightarrows}$
$\mathbf{R}_{\leftleftarrows} \equiv \{R_{i\leftleftarrows}\}_i$	Vector of each bank's total payment received	$\mathbf{R}_{\leftleftarrows} = \hat{\mathbf{F}}^\top \mathbf{R}_{\rightrightarrows}$
α	Scale of debts used in, e.g., Proposition 2	$\alpha > 0$
β	Bottleneck parameter (Definition 6)	
$d_{i\to j}$	Harmonic distance from B_i to B_j (Definition 5)	$d_{i\to j} \geq 0$
δ	Connectedness parameter (Definition 4)	$0 < \delta < 1$
d^{ST}, d^{LT}	Default and salvation radii in Lemma 4 and Proposition 4	
$\beta^{ST}, \beta_{ST}, \beta^{LT}, \beta_{LT}$	Thresholds in Lemma 5 and Proposition 5	
s	Dominance parameter (Definition 9)	$0 < s < 1$
s^*	Threshold in Proposition 6	
t_i	Transfer to B_i in Definition 7	
i^*	Index of largest liquidated bank in Section 5 and Section 6.3	
π_i	Permutation of banks keeping B_i fixed	
$[\cdot]^+ = \max\{\cdot, 0\}$	Maximum of variable and zero	
$\lceil\cdot\rceil, \lfloor\cdot\rfloor$	Ceiling and floor functions	
	Notations Used Only in Extensions	
y^*	Efficient liquidation threshold in Section 6.1	
Δ_i	Efficiency loss if B_i is liquidated in Section 6.3	
$\mathbf{x} \equiv \{x_i\}_i$	Vector of indicators of banks not being liquidated in Section 6.3	$\mathbf{x} \in \{0,1\}^N$
$\hat{\mathbf{t}}, \check{\mathbf{t}}, \hat{\mathbf{x}}, \check{\mathbf{x}}$	Optimizers in Section 6.3	
π	Ranking of banks for greedy algorithm in Section 6.3	
y_i	B_i's long-term real asset value in Section 6.3	$y_i > 0$
ℓ_i	Size of B_i's liquidity shock in Section 6.3	$\theta y_i < \ell_i < y_i$
	Notations Used Only in Proofs	
$\mathbf{Q}$	A matrix, usually shorthand for $\hat{\mathbf{F}}^\top$	
$\mathbf{Q}_{\mathscr{B}_1,\mathscr{B}_2} \equiv [Q_{ij}]_{i\in\mathscr{B}_1, j\in\mathscr{B}_2}$	Block matrix with rows in $\mathscr{B}_1$ and columns in $\mathscr{B}_2$	
$\tilde{\mathbf{Q}}$	Matrix in Lemma A.8	

Continued on next page

Notations (continued)

Notation	Meaning	Parametric restriction
$\mathbf{0}, \mathbf{1}$	Vectors of zeros and ones $((0, ..., 0)$ and $(1, ..., 1))$	
$\mathbf{d}_{i\to\mathscr{B}} \equiv \{d_{i\to j}\}_{j\in\mathscr{B}}$	Vector of B_i's harmonic distance to banks in $\mathscr{B}$	
$D_i \equiv F_{i\rightrightarrows} - R_{i\rightrightarrows}$	B_i's shortfall	
$\mathbf{D} \equiv \mathbf{F}_{\rightrightarrows} - \mathbf{R}_{\rightrightarrows}$	Vector of each bank's shortfall	
$\mathscr{D}$	Set of banks that default	
Φ^α, Ψ^α	Mappings used in Lemma A.1 and Lemma A.6	
$\mathscr{H}^\alpha, \mathscr{I}^\alpha$	Restricted domains of Φ^α and Ψ^α	
$\prod_{i=1}^N X_i = X_1 \times \cdots \times X_N$	Cartesian product of sets $X_1, ..., X_N$	

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